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USE OF SEQUENTIAL DIFFERENCES

IN SMOOTHING 3-D DATA

by

Joseph

J. Bryce Tysver

May 1979

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The use of sequential (successive) differences in detecting wild data or outliers in data collected on torpedo paths is explored. It is assumed that the collected data on torpedo paths is of the form

$$x_i = x(t_i) = P_x(t_i) + n_i + d_i$$

where $P_x(t_i)$ is the x component of the position of the torpedo at time t

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20. Abstract cont.

with P(t) being some appropriate low order polynomial, n_i is the 'noise' due to measurement error, and d_i is a perturbation or disturbance which if present in sufficient amplitude will cause x_i to be a 'wild' datum or outlier.

Variations in the patterns (signatures) of successive differences caused by a variety of perturbations are examined for the purpose of setting thresholds to be used to detect outliers. Data collected from a torpedo path at NUWES are used for illustration.

USE OF SEQUENTIAL DIFFERENCES IN SMOOTHING 3-D DATA

by

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1. INTRODUCTION

In a previous report (Reference 1), the author proposed the use of sequential (successive) differences as an aid in identifying outlier data points and in selecting the appropriate order polynomial for smoothing of 3-D data on torpedo and target paths. In this report, the concept of successive differences is explored and developed with the specific intent of making it suitable for inclusion in a computer program for smoothing 3-D data.

The nature of the report is in the form of a working paper rather than a polished formal report. Some of the discussions presented are rather lengthy and points of interest are, perhaps, belabored and/or repeated unnecessarily. The reader's indulgence is invited and some skimming is expected. Nevertheless the general picture appears clear and the possibility of using the model for identification of outliers reasonable.

2. DEVELOPMENT OF MODEL

A. General Considerations

For the purposes of this analysis, it will be assumed that an observed datum x_i can be expressed in the form

$$x_i = x(t_i) = P_x(t_i) + n_i + d_i$$

where $P(t)$ is a polynomial in time t, n_i is a measurement error which will be called "noise," and d_i is a perturbation or disturbance which, if present with sufficient amplitude, will cause x_i to be a "wild" datum or outlier.

It will be assumed that each component (x,y,z) of a torpedo (T) or target (submarine, S) path can be represented as a polynomial of some low degree k in time t. (It is suggested that the restriction $k \leq 4$ be incorporated in the smoothing algorithm.) Thus

$$P_x(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_k t^k.$$

The noise component, n_i, is assumed to be a realization of a random variable N_i which is Normally distributed with mean 0 and common variance σ^2 $(N_i \sim N(0,\sigma^2))$ and it is also assumed that noise components N_i and N_j at times t_i and t_j are independent.

Finally, it will be assumed that a disturbance d_i should have fairly rare occurrence. Evidence of the existence of a non-zero value of d_i can be obtained from examination of successive differences which, when sufficiently high order differences are considered, are functions of the $(n_i + d_i)$'s and not of the $P(t_i)$'s. Crossing of a threshold value for successive differences, which is seldom crossed when no d_i's are present, can then be used as an indication of the presence of a disturbance d_i and hence of an outlier point. Note that, not only can noise only cause an occasional crossing depending on the threshold selected, but the presence of a disturbance may not cause a threshold crossing depending on its magnitude and its interaction with noise. This will be elaborated as the development of the model progresses.

B. <u>Successive Differences</u>

A definition of successive or sequential differences suitable for our purposes is presented in the accompanying table (Table 1) and the notation which follows. Since the 3-D data to be smoothed involves data points equally spaced in time, this has been incorporated in the model. Further, the initial time for any data segment can be arbitrarily set to zero for model development hence $t_0 = 0$. Also, selection of the common time interval as the unit of time yields

$$t_{i+1} = t_i + 1.$$

TABLE 1

SUCCESSIVE DIFFERENCES

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$P(0) + n_0$				
		$D_{11} = P_{11} + n_{11}$			
1	$P(1) + n_1$		$D_{21} = P_{21} + n_{21}$		
		$D_{12} = P_{12} + n_{12}$		$D_{32} = P_{32} + n_{32}$	
2	$P(2) + n_2$		$D_{22} = P_{22} + n_{22}$		$D_{42} = P_{42} + n_{42}$
		$D_{13} = P_{13} + n_{13}$		$D_{33} = P_{33} + n_{33}$	
3	$P(3) + n_3$		$D_{23} = P_{23} + n_{23}$		$D_{43} = P_{43} + n_{43}$
		$D_{14} = P_{14} + n_{14}$		$D_{34} = P_{34} + n_{34}$	
4	$P(4) + n_4$		$D_{24} = P_{24} + n_{24}$		$D_{44} = P_{44} + n_{44}$
		$D_{15} = P_{15} + n_{15}$		$D_{35} = P_{35} + n_{35}$	
5	$P(5) + n_5$		$D_{25} = P_{25} + n_{25}$		
		$D_{16} = P_{16} + n_{16}$			
6	$P(6) + n_6$				

$D_{i1} = x_i - x_{i-1}$ | $P_{1i} = P(i) = P(i-1)$ | $n_{1i} = n_i - n_{i-1}$

$D_{2i} = D_{1,i+1} - D_{1i}$ | $P_{2i} = P_{1,i+1} - P_{1i}$ | $n_{2i} = n_{1,i+1} - n_{1i}$

$D_{3i} = D_{2i} - D_{2,i-1}$ | $P_{3i} = P_{2i} - P_{2,i-1}$ | $n_{3i} = n_{2i} - n_{2,i-1}$

$D_{4i} = D_{3,i+1} - D_{3i}$ | $P_{4i} = P_{3,i+1} - P_{3i}$ | $n_{4i} = n_{3,i+1} - n_{3i}$

The selection of the secondary subscript i in the ordered differences is somewhat arbitrary. As will be noted when disturbances are introduced, it appears desirable for computational convenience to identify the even ordered differences (D_{2i} and D_{4i}) with the observation x_i for each i. For example, a large isolated disturbance d_i in x_i will produce large perturbations in D_{2i} and D_{4i} hence the latter can be used to identify x_i as an 'outlier.' For the odd ordered differences (D_{1i} and D_{3i}) the situation is not as clear. For example, if a large perturbation is observed in D_{3i} it is not clearly evident whether x_i or x_{i-1} should be considered as the 'outlier.' At this stage in the development, it would appear that the even ordered successive differences should be the primary identifiers of 'outliers.'

C. The Polynomial Component

To illustrate the contribution of the polynomial component to successive differences, three cases (linear, quadratic, and cubic) polynomials are presented in Tables 2.1, 2.2 and 2.3. It can readily be seen that there is a contribution of a polynomial of degree k to D_{ji} for $j \leq k$ but that for $j > k$ the number D_{ji} represents noise only unless a disturbance is present. Thus detection of a disturbance, and hence identification of an outlier, becomes simpler if a sufficiently high order difference can be used and the polynomial component eliminated.

TABLE 2.1. SUCCESSIVE DIFFERENCES

Linear Case: $x_i = x(t_i) = a_0 + a_1 t_i + n_i$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$D_{11} = a_1 + N_{11}$			
1	$a_0 + a_1 + n_1$		$D_{21} = n_{21}$		
		$D_{12} = a_1 + N_{12}$		$D_{32} = n_{32}$	
2	$a_0 + 2a_1 + n_2$		$D_{22} = n_{22}$		$D_{42} = n_{42}$
		$D_{13} = a_1 + N_{13}$		$D_{33} = n_{33}$	
3	$a_0 + 3a_1 + n_3$		$D_{23} = n_{23}$		$D_{43} = n_{43}$
		$D_{14} = a_1 + N_{14}$		$D_{34} = n_{34}$	
4	$a_0 + 4a_1 + n_4$		$D_{24} = n_{24}$		$D_{44} = n_{44}$
		$D_{15} = a_1 + N_{15}$		$D_{35} = n_{35}$	
5	$a_0 + 5a_1 + n_5$		$D_{25} = n_{25}$		
		$D_{16} = a_1 + N_{16}$			
6	$a_0 + 6a_1 + n_6$				

TABLE 2.2

Quadratic Case: $x_i = x(t_i) = a_0 + a_1 t_i + a_2 t_i^2 + n_i$

$t_0 = 0,\ t_{i+1} = t_i + 1,\ n_i \sim N(0,\sigma^2)$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + a_2 + n_{11}$			
1	$a_0 + a_1 + a_2 + n_1$		$2a_2 + n_{21}$		
		$a_1 + 3a_2 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + 4a_2 + n_2$		$2a_2 + n_{22}$		n_{42}
		$a_1 + 5a_2 + n_{13}$		n_{33}	
	$a_0 + 3a_1 + 9a_2 + n_3$		$2a_2 + n_{23}$		n_{43}
		$a_1 + 7a_2 + n_{14}$		n_{34}	
4	$a_0 + 4a_1 + 16a_2 + n_4$		$2a_2 + n_{24}$		n_{44}
		$a_1 + 9a_2 + n_{15}$		n_{35}	
5	$a_0 + 5a_1 + 25a_2 + n_5$		$2a_2 + n_{25}$		
		$a_1 + 11a_2 + n_{16}$			
6	$a_0 + 6a_1 + 36a_2 + n_6$				

TABLE 2.3

Cubic Case: $x_i = x(t_i) = a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3 + n_i$

$t_0 = 0, \quad t_{i+1} = t_i + 1, \quad n_i \sim N(0,\sigma^2)$

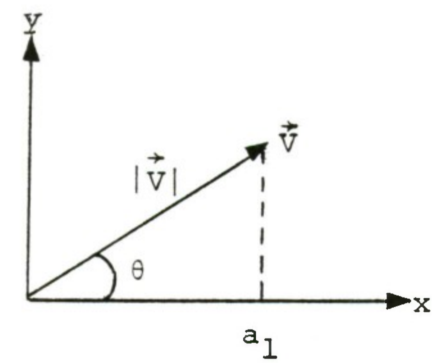
t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + a_2 + a_3 + n_{11}$			
1	$a_0 + a_1 + a_2 + a_3 + n_1$		$2a_2 + 6a_3 + n_{21}$		
		$a_1 + 3a_2 + 7a_3 + n_{12}$		$6a_3 + n_{32}$	
2	$a_0 + 2a_1 + 4a_2 + 8a_3 + n_2$		$2a_2 + 12a_3 + n_{22}$		n_{42}
		$a_1 + 5a_2 + 19a_3 + n_{13}$		$6a_3 + n_{33}$	
3	$a_0 + 3a_1 + 9a_2 + 27a_3 + n_3$		$2a_2 + 18a_3 + n_{23}$		n_{43}
		$a_1 + 7a_2 + 37a_3 + n_{14}$		$6a_3 + n_{34}$	
4	$a_0 + 4a_1 + 16a_2 + 64a_3 + n_4$		$2a_2 + 24a_3 + n_{24}$		n_{44}
		$a_1 + 9a_2 + 61a_3 + n_{15}$		$6a_3 + n_{35}$	
5	$a_0 + 5a_1 + 25a_2 + 125a_3 + n_5$		$2a_2 + 30a_3 + n_{25}$		
		$a_1 + 11a_2 + 91a_3 + n_{16}$			
6	$a_0 + 6a_1 + 36a_2 + 216a_3 + n_6$				

The question of how high the order of the difference must be to eliminate the polynomial component is not clear-cut. As a matter of fact, the polynomial component does not have to be eliminated entirely for a particular order of successive differences to be used to identify outliers. It is sufficient that the contribution of the polynomial component P_{ji} be small with respect to the noise component N_{ji} for D_{ji} to be useful as an indicator of a disturbance d_i in x_i.

(This is intimately related to the problem of fitting polynomials to segments of a torpedo path. If (1) torpedo path does not change too radically, (2) the length of the path segment to be fitted is short enough, and (3) the data rate is high enough, then low order polynomials can provide satisfactory approximations to the path. In Reference 1, path segments of 21 and 11 points were explored briefly. Path segments consisting of 7 points has been suggested but not examined as yet. In many of these segments examined polynomials of order $k \leq 3$ produced acceptably small and apparently random residual errors for 11 point segments.)

From Tables 1, 2.1-2.3 it can be seen that a successive difference D_{ji} of order j involves $j+1$ successive observations x_i. For $j \leq 4$, as proposed for screening for outliers, at most five data points are involved. These can be fitted reasonably well by polynomials of order $k \leq 3$. Supporting evidence for this is available in the successive differences for the 3-D data on the torpedo run examined in this study. Discussion of the analysis justifying this contention will be presented in a later section.

An alternative has been suggested. It incorporates control information (information obtained by alternate means on the command and control of a torpedo) to provide appropriate values for the polynomial coefficients and to indicate appropriate polynomial order for fitting data. In the linear case this information should be in the form of a specific value or bound for a_1. Since $a_1 = |\vec{V}| \cos \theta$, as illustrated in the accompanying sketch with $\vec{V}$ a velocity vector and $|\vec{V}|$ the magnitude of $\vec{V}$, one possible value for a_1 would be $a_1 \leq |V|$.



This will be shown to dominate the noise component N_{1i} for 3-D data. Information from control data on θ could be used but would require a_1 (and hence the threshold D_1^*) to be treated as a function of position on the torpedo path and hence as a function of t_i. For the purpose of preliminary screening for outliers, it would appear preferable to concentrate on successive differences of sufficiently high order that the polynomial component can be considered negligible. With this constraint, a constant threshold D_j^* can be used for all successive differences D_{ji} of order j.

D. The Noise Component

When the polynomial component P_{ji} has been eliminated, attention can be concentrated on the noise component n_{ji} of the jth order successive differences. In engineering parlance, the problem of identifying outliers can now be considered as one of detecting a signal (a disturbance d_i) in the presence of noise (n_{ji}). The thresholds D_j^* can be expressed as specified levels of D_{ji} which are seldom exceeded by noise only and hence which indicate the presence of a disturbance d_i. In order to establish values for D_j^*, a statistical analysis of the noise component is required.

Recall the assumptions in Section 2.A that the noise component n_i is a realization of a random variable N_i with $N_i \sim N(0,\sigma^2)$ and that N_i and N_j are independent for $i \neq j$. It can be established from the definitions of successive differences that the noise component N_{ji} of D_{ji} can be defined in terms of the noise components n_i of x_i as follows:

$$n_{1i} = n_i - n_{i-1}$$

$$n_{2i} = n_{i+1} - 2n_i + n_{i-1}$$

$$n_{3i} = n_{i+1} - 3n_i + 3n_{i-1} - n_{i-2}$$

$$n_{4i} = n_{i+2} - 4n_{i+1} + 6n_i - 4n_{i-1} + n_{i-2} \text{ .}$$

Each of these noise components have mean 0 since the n_i's are assumed to have mean 0.

The variance V_j of N_{ji} can be expressed in terms of the common variance σ^2 of the n_i's using the independence property of the n_i's. These are presented below together with some of the covariances $C(n_{ji}, k_{kr})$ of interest later.

1^{st} Order Noise Differences (N_{1i})

$$V_1 = 2\sigma^2$$

$$C(n_{1i}, n_{1,i+1}) = -\sigma^2$$

2^{nd} Order Noise Differences (N_{2i})

$$V_2 = 6\sigma^2$$

$$C(n_{2i}, n_{2,i+1}) = -4\sigma^2$$

$$C(n_{2i}, n_{2,i+2}) = \sigma^2$$

3^{rd} Order Noise Differences (N_{3i})

$$V_3 = 20\sigma^2$$

$$C(n_{3i}, n_{3,i+1}) = -12\sigma^2$$

$$C(n_{3i}, n_{3,i+2}) = 6\sigma^2$$

4^{th} Order Noise Differences (N_{4i})

$$V_4 = 70\sigma^2$$

$$C(n_{4i}, n_{4,i+1}) = -56\sigma^2$$

$$C(n_{4i}, n_{4,i+2}) = 28\sigma^2$$

Selected Covariances

$$C(n_{2i}, n_{3i}) = 10\sigma^2$$

$$C(n_{2i}, n_{3,i+1}) = -10\sigma^2$$

$$C(n_{2i}, n_{4i}) = -20\sigma^2$$

$$C(n_{3i}, n_{4,i}) = -35\sigma^2$$

$$C(n_{3,i+1}, n_{4i}) = 35\sigma^2$$

Since all the N_{ji}'s are normally distributed with mean 0, it can be established that

$$P(|N_{ji}| \geq 3\sqrt{V_j}) \doteq 0.997 .$$

If we set $D_j^* = 3\sqrt{V_j}$ then, for applications in which the polynomial contributions to D_{ji} have been eliminated, there will be, on the average, less than one time in 200 independent trials in which the $|D_{ji}|$ will exceed D_j^* due to noise alone. The suggested thresholds for detection of disturbances are given below.

j	0	1	2	3	4
D_j^*	3σ	4.24σ	7.348σ	13.416σ	25.10σ

The term D_j^* with $j = 0$ corresponds to $V_i = \sigma^2$ (i.e., the variance of N_i and hence of x_i when no polynomial is involved).

The suggested thresholds are worth some further exploration. As an oversimplified case consider a situation in which no polynomial contributions are involved, $n_k = 3\sigma$ for some k, and $n_i = 0$ for $i \neq k$. The relationships of the D_{jk}'s to the D_j^*'s are shown in the following table.

j	0	1	2	3	4
$D_{jk} = n_{jk}$	3σ	3σ	-6σ	-9σ	18σ
D_k^*	3σ	4.24σ	7.35σ	13.4σ	25.1σ
$\lvert n_{1k}\rvert / D_j^*$	1	.707	.816	.671	.717

Since $\lvert n_{2k}\rvert / D_2^*$ is greater than the corresponding expression for $j = 3$ or $j = 4$, it could be anticipated that the second order differences (the D_{2i}'s) might be better detectors for disturbances when the polynomial contribution is linear. This will be demonstrated for an isolated disturbance in a later section of this report.

The type of information to be seen in the special case of an isolated noise element n_k can be generalized. The covariances are useful for this purpose. Note that, comparing the special case to the covariances,

Special Case	Covariance
$D_{2k} = -6\sigma$, $D_{2,k\pm1} = 3\sigma$	$C(n_{2i}, n_{3i\pm1}) = -4\sigma^2$
$D_{2k} = -6\sigma$, $D_{3k} = -9\sigma$	$C(n_{2i}, n_{3i}) = +10\sigma^2$
$D_{2k} = -6\sigma$, $D_{4k} = 18\sigma$	$C(n_{2i}, n_{4i}) = -20\sigma^2$

This relationship can, perhaps, be made clearer by considering the correlation coefficients. For example,

$$r(n_{2i}, n_{4i}) \equiv \frac{C(n_{2i}, n_{4i})}{\sqrt{V_2V_4}} = \frac{-20\sigma^2}{\sqrt{(6\sigma^2)(70\sigma^2)}} = -0.976$$

The other correlation coefficients of interest here are

$$r(n_{2i}, n_{2,i+1}) = \frac{-4}{6} = -0.667 \ ,$$

$$r(n_{2i}, n_{3i}) = \frac{10}{\sqrt{120}} = 0.913 \ ,$$

$$r(n_{3i}, n_{3,i+1}) = \frac{-12}{20} = -0.6 \ ,$$

$$r(n_{3i}, n_{4i}) = \frac{-35}{\sqrt{1400}} = -0.935 \ ,$$

and

$$r(n_{4i}, n_{4,i+1}) = \frac{-56}{70} = -0.8 \ .$$

These can be interpreted as follows. In general, if n_{2i} has a large value, then $n_{2,i+1}$ and n_{4i} can be expected to have fairly large values of the opposite sign and n_{3i} a fairly large value of the same sign. The importance of this in detecting outliers is that the information provided by different orders of differences at the same point and by differences of the same order at adjacent points is primarily of a confirmation nature rather than providing complementary information. This can be interpreted to the more practical statement that, for example, if a disturbance in x_i which does not cause a crossing of D_4^* by D_{4i}, then it will usually not cause a threshold crossing by D_{2i}, D_{3i}, $D_{4,i-1}$ or $D_{4,i+1}$. On the other hand, if D_{4i} exceeds D_4^* in magnitude, then one or more of these other differences has a reasonable chance of crossing its prescribed threshold.

As a consequence of the complementary nature of threshold crossings and of the fact that D_{4i} is less likely to be contaminated by a polynomial component, it is suggested that the testing for outliers be performed by testing only fourth order differences (the D_{4i}'s) for crossing of the appropriate threshold D_4^*.

Before considering the disturbance component of x_i, it would be of interest to consider the relative magnitudes of polynomial and noise components of 3-D data. Of particular interest here is the comparison of a_1 with D_1^* since these are the vital components if the first order differences are to be used for detecting outliers. Since $a_1 = |\vec{V}| \cos \theta$, it can be seen that a_1 achieves its maximum magnitude when $\theta = 0°$ or $\theta = 180°$. A plot of the path of the torpedo in the torpedo run selected for examination in this study and the corresponding data together with the first four orders of differences are presented in Appendix A. It can be seen that $\theta = 0°$ occurs in the vicinity of $t = 950$ and $\theta = 180°$ occurs in the vicinities of $t = 807$, 853, and 917. An approximate value of $|\vec{V}|$ is satisfactory for the present purposes and the value $|\vec{V}| = 95$ will be used.

Establishment of a bound for the noise in the form with

$$P(|N_{1i}| > 3\sigma_{N_1}) < 0.01 ,$$

with $\sigma_{N_1}^2 = 2\sigma^2$, requires estimation of σ^2, the noise variance. In Reference 1, estimates of σ as low as 2 or 3 were obtained for selected segments of the torpedo run to be used here. It will be assumed for this examination that $\sigma = 4$ and hence that $\sigma_{N_1} \doteq 5.656$ and hence $3\sigma_{N_1} \doteq 17$.

Boundary for D_{1i} can then be set in the form

$$D_1^* = \pm [|\vec{V}| + 3\sigma_{N_1}] = \pm 112.$$

Thus, only if D_{1j} were greater than + 112 or less than -112 would a disturbance be indicated. Using the formula

$$D_j^* = |\vec{V}| \cos \theta \pm 3\sigma_{N_1}$$

when θ is given we have

θ	Δ_1^*	
	Lower threshold	Upper threshold
0°	$95 - 17 = 78$	$95 + 17 = 112$
90°	-17	$+17$
180°	$-95 - 17 = -112$	$-95 + 17 = -78$

It can be seen that detection of disturbances in the first order differences unless $\theta = 0^\circ$ or 180° will not be reliable when a general threshold of the form

$$D_1^* = \pm [|\vec{V}| + 3\sigma_{n_1}]$$

is used.

E. The Disturbance Component

The presence of a disturbance or perturbation in an observation x_i can be represented as an additional component d_i so that

$$x_i = x(t_i) = P(t_i) + n_i + d_i \ .$$

There are several types of perturbations that could be considered. One of these, an 'outlier' or isolated disturbance d_i that occurs in only one observation x_i, is the simplest. The effects of such a disturbance is shown in Table 3.1 and the accompanying sketch, Figure 3.1. In the sketch both d and the D_j^*'s are expressed in terms of the parameter σ (the standard deviation of the noise component n_i). The value $d = 5\sigma$ is used for illustrative purposes. Also note that the ordinate is

$$x'_{ji} = x_{ji} - P_{ji} - n_{ji} = d_{ji}$$

and hence represents only the disturbance component of x_{ji}.

There are several features of the successive differences that should be noted when an isolated perturbation occurs. First, consider an observation x_i (in our example $x_i = 4$) consisting of an isolated disturbance $d = k\sigma$ without any noise ($n_{ji} = 0$ for all j and i) and with polynomial component $P(t_i) = a_0 + a_1 t_i$. The values of k for which the thresholds (D_j^*'s) are achieved are shown below.

j	2	3	4
D_{j4}	$2k\sigma$	$3k\sigma$	$6k\sigma$
D_j^*	7.35σ	13.4σ	25.1σ
Critical k	3.675	4.467	4.183

In the absence of the noise and polynomial components, the second order difference D_{2i} will provide a threshold crossing for a smaller isolated disturbance ($d \geq 3.675\sigma$) than either the third order difference $(d \geq 4.476\sigma)$ or the fourth order difference $(d \geq 4.183\sigma)$ and D_{4i} is slightly better than D_{3i}. If assurances could be given that the polynomial component was no higher than the first degree, then the second order differences (the D_{2i}'s) would appear to provide the most sensitive location to test for isolated disturbances. If polynomial components of the second or third degrees are possible then the fourth order differences (the D_{4i}'s) appear to be preferable for testing.

Next, consider the pattern or signature produced in the ordered differences by an isolated disturbance at t_r. Both D_{2i} and D_{4i} will contain their maximum contributions from the disturbance at D_{2r} and D_{4r} (of opposite signs) and both will have substantial but smaller contributions of opposite signs at the adjacent points ($D_{2,r-1}$ and $D_{2,r+1}$ and $D_{4,r-1}$ and $D_{4,r+1}$). The third order differences (the D_{3i}'s) will have contributions of equal magnitudes but opposite signs at adjacent positions ($D_{3,r}$ and $D_{3,r+1}$) and smaller contributions at the next positions. Incorporation of their signatures, although clearly recognizable, in the graph (see Fig. 3.1) would be difficult to incorporate in a program for automatic computer filtering of outliers.

The last item for discussion of isolated disturbances pertains to the addition of noise and disturbance components. Consider, now a disturbance $d = 5\sigma$ in x_r (x_4 in Table 3.1) and its effect on D_{4r} in the presence of noise. A positive value of n_{4r} will enhance crossing the threshold D_4^* so attention can be directed to the effects of negative values for n_{4r}. If

$$n_{4r} < -(30\sigma - 25.1\sigma) = -4.9\sigma\left(\frac{\sigma_{n_{4i}}}{\sqrt{70}\ \sigma}\right) = -.586\sigma_{n_{4i}}$$

then D_{4r} will not cross the upper threshold $D_4^* = 25.1\sigma$. For this situation the probability of a threshold crossing is $P(N_{4n} > D_4^*) = .721$. In this event $n_{4,r-1}$ and $n_{4,r+1}$ will, in general, be positive since

$$r(n_{4i}, n_{4,i-1}) = r_i(n_{4i}, n_{4,i+1}) = -0.8 \quad \text{(Section C)}$$

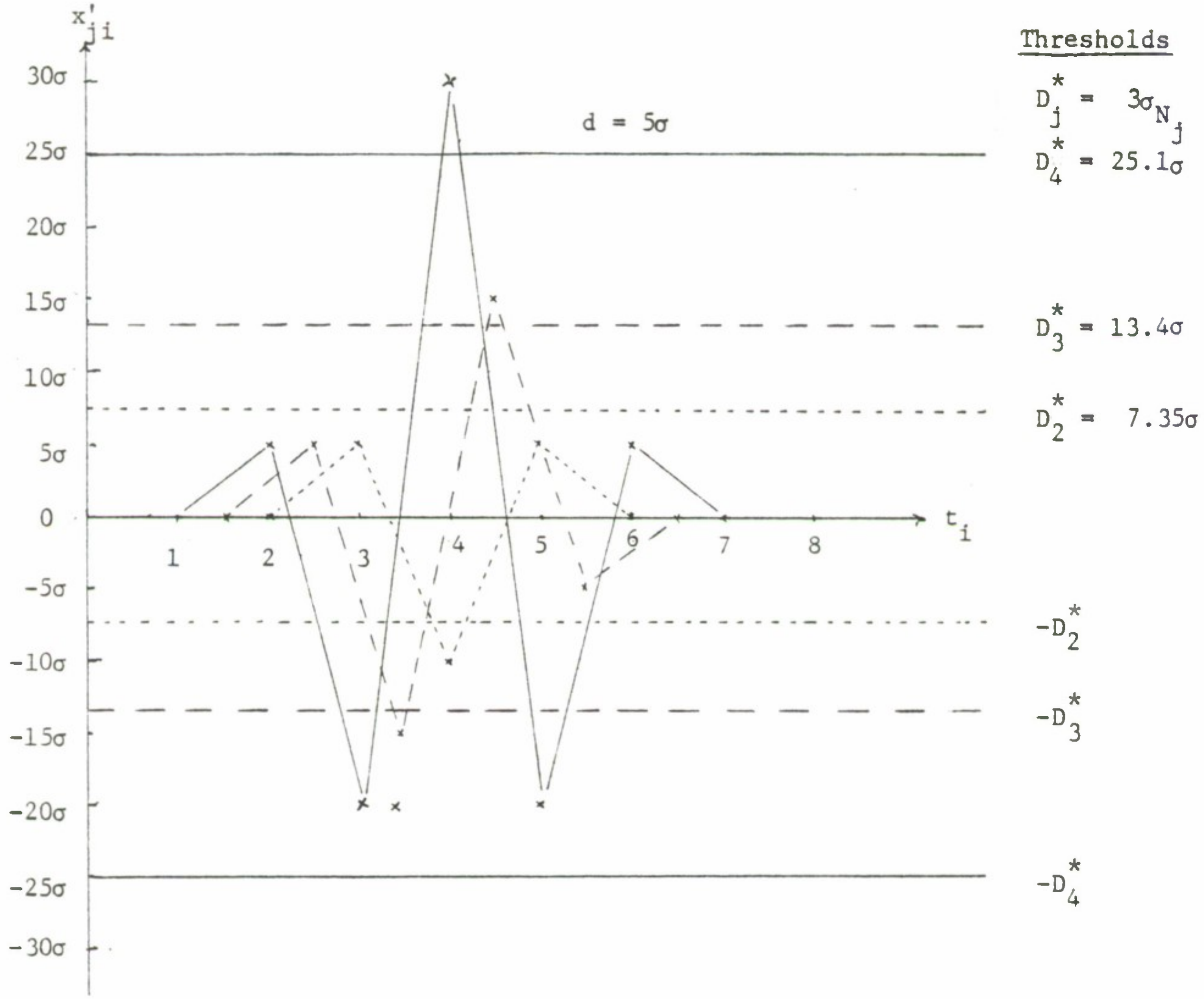
and hence neither $D_{4,r-1}$ nor $D_{4,r+1}$ can be expected to cross the lower threshold $D_4^* = -25.1\sigma$. Also, as a consequence of $r(n_{2i}, n_{4i}) = -0.976$, a negative value for n_{4r} can be expected to be accompanied by a positive value for n_{2r} and hence D_{2r} will not cross the lower threshold $D_2^* = -7.35\sigma$. Further, since $r(n_{2i}, n_{2,i+1}) = -0.667$, neither $D_{2,r-1}$ nor $D_{2,r+1}$ can be expected to cross the upper threshold $D_2^* = +7.35\sigma$. Similarly the correlations $r(n_{3i}, n_{4i}) = -0.935$ and $r(n_{3i}, n_{3,i+1}) = -0.6$ make it unlikely that either D_{3r} or $D_{3,r+1}$ will cross the lower threshold $D_3^* = -13.4\sigma$ or the upper threshold $D_3^* = +13.4\sigma$, respectively.

TABLE 3.1

SUCCESSIVE DIFFERENCES

Linear Case: Isolated Disturbance d

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		$n_{42} + d$
		$a_1 + n_{13}$		$n_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$n_{23} + d$		$n_{43} - 4d$
		$a_1 + n_{14} + d$		$n_{34} - 3d$	
4	$a_0 + 4a_1 + n_4 + d$		$n_{24} - 2d$		$n_{44} + 6d$
		$a_1 + n_{15} - d$		$n_{35} + 3d$	
5	$a_0 + 5a_1 + n_5$		$n_{25} + d$		$n_{45} - 4d$
		$a_1 - n_{16}$		$n_{36} - d$	
6	$a_0 + 6a_1 + n_6$		n_{26}		$n_{46} + d$
		$a_1 + n_{17}$		n_{37}	
7	$a_0 + 7a_1 + n_7$		n_{27}		
		$a_1 + n_{18}$			
8	$a_0 + 8a_1 + n_8$				



x'ji
30σ
25σ
20σ
15σ
10σ
5σ
0
-5σ
-10σ
-15σ
-20σ
-25σ
-30σ
d = 5σ
1
2
3
4
5
6
7
8
ti
Thresholds
D*j = 3σNj
D*4 = 25.1σ
D*3 = 13.4σ
D*2 = 7.35σ
-D*2
-D*3
-D*4

FIGURE 3.1

The proposed use of only one order of successive difference (namely, D_{4i}) to test for outliers appears reasonable for isolated disturbances. If D_{4r} exceeds its threshold then this will will usually be accompanied by $D_{2,r}$ and $D_{3,r}$ exceeding their thresholds in the opposite direction.

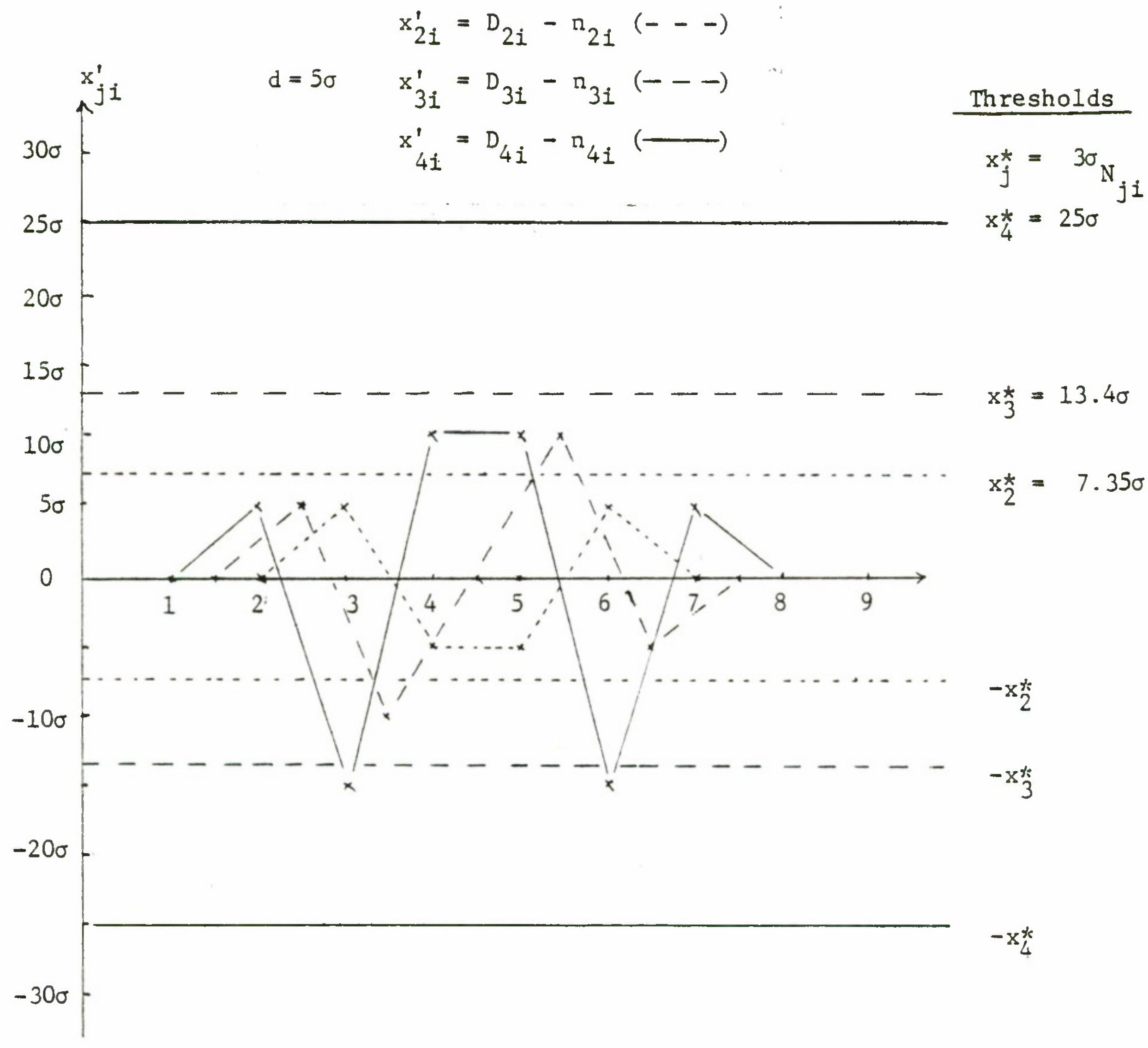
Attention can now be directed to disturbances other than isolated ones. Consider, next, a situation involving disturbances d_i and d_k in two observations. For simplicity, it will be assumed that they have the same magnitude, d, but can differ in sign and/or location. The situation with two adjacent disturbances of the same sign is presented in Table 3.2 and Figure 3.2. Note that the magnitudes of the contributions of the disturbances to D_{44} and D_{45} (D_{4r} and $D_{4,r+1}$ for equal disturbances in x_r and x_{r+1}) is substantially reduced from that in case of an isolated disturbance as is the contributions to the next adjacent observations. It is evident that large adjacent disturbances of the same sign will be less likely to cause threshold crossings. Note that a large noise component in one observation (n_{4r}, for example) will, in general, be accompanied by a large noise component of the opposite sign ($r(n_{4i}, n_{4,i+1}) = -0.8$) in the other observation and hence enhance the probability of a threshold crossing by one of the differences D_{4r} or $D_{4,r+1}$. In general, two adjacent large values of the same sign in D_{2i} or D_{4i} is a signature

TABLE 3.2

SUCCESSIVE DIFFERENCES

Linear Case: Adjacent Equal Disturbances

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		$n_{42} + d$
		$a_1 + n_{13}$		$n_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$n_{23} + d$		$n_{43} - 3d$
		$a_1 - n_{14} + d$		$n_{34} - 2d$	
4	$a_0 + 4a_1 + n_4 + d$		$n_{24} - d$		$n_{44} + 2d$
		$a_1 + n_{15}$		n_{35}	
5	$a_0 + 5a_1 + n_5 + d$		$n_{25} - d$		$n_{45} + 2d$
		$a_1 + n_{16} - d$		$n_{36} + 2d$	
6	$a_0 + 6a_1 + n_6$		$n_{26} + d$		$n_{46} - 3d$
		$a_1 + n_{17}$		$n_{37} - d$	
7	$a_0 + 7a_1 + n_7$		n_{27}		$n_{47} + d$
		$a_1 + n_{18}$		n_{38}	
8	$a_0 + 8a_1 + n_8$		n_{28}		
		$a_1 + n_{19}$			
9	$a_0 + 9a_1 + n_9$				



x'2i = D2i - n2i (- - -)
d = 5σ
x'3i = D3i - n3i (— — —)
x'4i = D4i - n4i (———)
x'ji
Thresholds
x*j = 3σNji
x*4 = 25σ
x*3 = 13.4σ
x*2 = 7.35σ
-x*2
-x*3
-x*4
30σ
25σ
20σ
15σ
10σ
5σ
0
-10σ
-20σ
-30σ
1
2
3
4
5
6
7
8
9

FIGURE 3.2

of adjacent disturbances of the same sign. (The possibility of using reduced thresholds for this situation has not been explored.) The magnitudes of the D_{3i}'s are also smaller than in the single disturbance situation and are separated by an observation (D_{35}) involving noise only.

Next, consider adjacent disturbances of equal magnitudes but opposite signs. This situation is presented in Table 3.3 and Figure 3.3. The additive, or magnification, effect of the opposing signs should make even moderate magnitudes of the disturbances readily detectable. The pattern or signature should be clearly evident. It is suspected, however, that the occurrence of this situation in real-life data would be extremely rare in comparison to the previous situation.

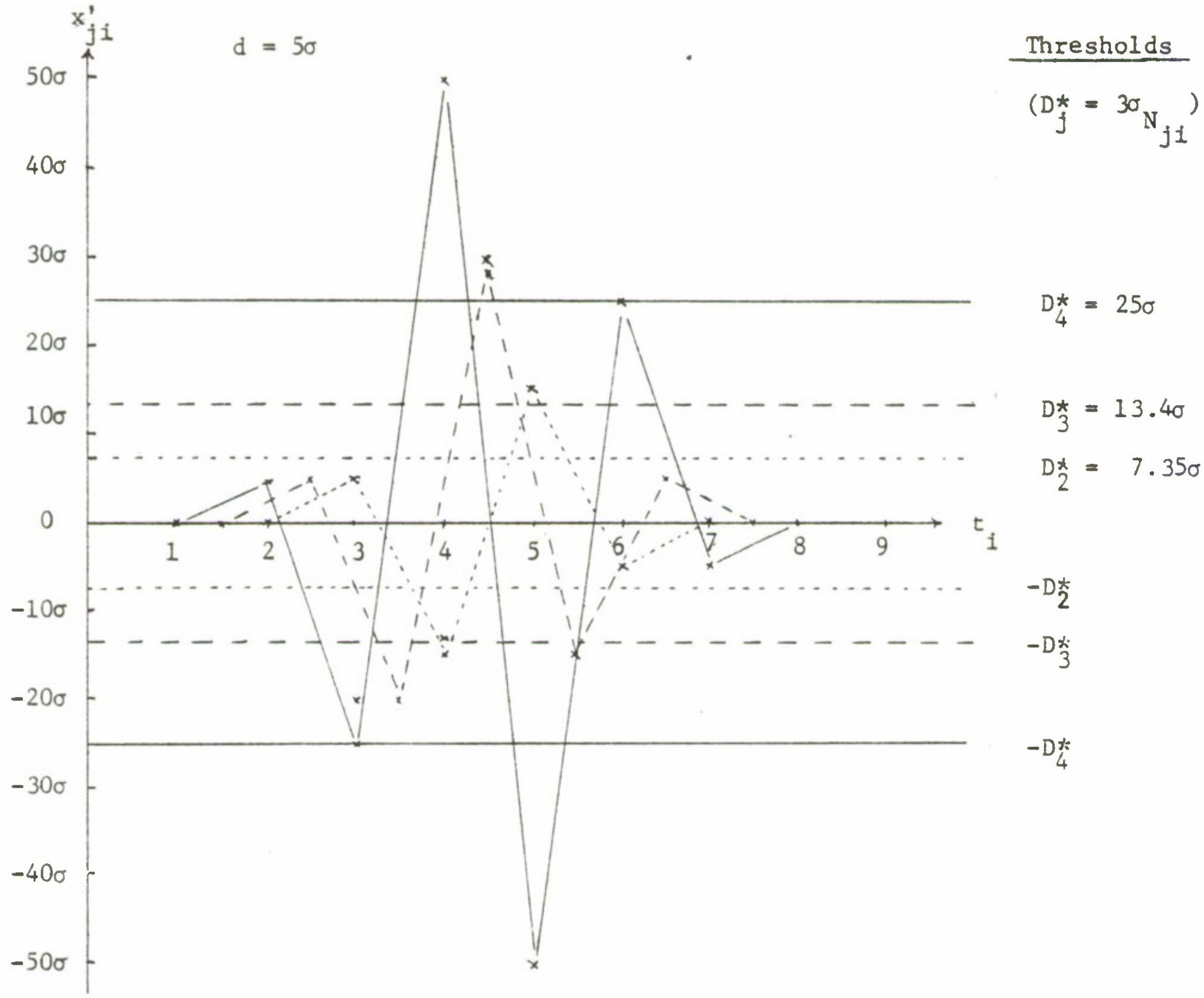
The situation in which two disturbances of similar magnitude and sign separated by one unperturbed data point is presented in Table 3.4 and Figure 3.4. From the graph it can be seen that this situation looks much like a situation with a single isolated disturbance of somewhat greater magnitude and opposite sign (Fig. 3.1). This brings the danger that the observation x_5 (between the two observations with disturbances) could be erroneously labeled as an outlier and hence removed and treated as a missing point. In the next section missing points and their replacement by the average of the observations on each side of the missing point will be discussed.

TABLE 3.3

SUCCESSIVE DIFFERENCES

Linear Case: Adjacent Opposed Equal Disturbances

t_i	x_i	D_{i1}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		$n_{42} + d$
		$a_1 + n_{13}$		$n_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$n_{23} + d$		$n_{43} - 5d$
		$a_1 + n_{14} + d$		$n_{34} - 4d$	
4	$a_0 + 4a_1 + n_4 + d$		$n_{24} - 3d$		$n_{44} + 10d$
		$a_1 + n_{15} - 2d$		$n_{35} + 6d$	
5	$a_0 + 5a_1 + n_5 - d$		$n_{25} + 3d$		$n_{45} - 10d$
		$a_1 + n_{16} + d$		$n_{36} - 4d$	
6	$a_0 + 6a_1 + n_6$		$n_{26} - d$		$n_{46} + 5d$
		$a_1 + n_{17}$		$n_{37} + d$	
7	$a_0 + 7a_1 + n_7$		n_{27}		$n_{47} - d$
		$a_1 + n_{18}$		n_{38}	
8	$a_0 + 8a_1 + n_8$		n_{28}		
		$a_1 + n_{19}$			
9	$a_0 + 9a_1 + n_8$				



x'_{ji}
d = 5σ
Thresholds
($D^*_j = 3\sigma_{N_{ji}}$)
50σ
40σ
30σ
20σ
10σ
0
-10σ
-20σ
-30σ
-40σ
-50σ
1
2
3
4
5
6
7
8
9
t_i
$D^*_4 = 25\sigma$
$D^*_3 = 13.4\sigma$
$D^*_2 = 7.35\sigma$
$-D^*_2$
$-D^*_3$
$-D^*_4$

FIGURE 3.3

TABLE 3.4

SUCCESSIVE DIFFERENCES

Linear Case: Two Disturbances Separated by One Point

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + N_{11}$			
1	$a_0 + a_1 + n_1$		N_{21}		
		$a_1 + N_{21}$		N_{32}	
2	$a_0 + 2a_1 + n_2$		N_{22}		$N_{42} + d$
		$a_1 + N_{13}$		$N_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$N_{23} + d$		$N_{43} - 4d$
		$a_1 + N_{14} - d$		$N_{34} - 3d$	
4	$a_0 + 4a_1 + n_4 + d$		$N_{24} - 2d$		$N_{44} + 7d$
		$a_1 + N_{15} - d$		$N_{35} + 4d$	
5	$a_0 + 5a_1 + n_5$		$N_{25} + 2d$		$N_{45} - 8d$
		$a_1 + N_{16} + d$		$N_{36} - 4d$	
6	$a_0 + 6a_1 + n_6 + d$		$N_{26} - 2d$		$N_{46} + 7d$
		$a_1 + N_{17} - d$		$N_{37} + 3d$	
7	$a_0 + 7a_1 + n_7$		$N_{27} + d$		$N_{48} - 4d$
		$a_1 + N_{18}$		$N_{38} - d$	
8	$a_0 + 8a_1 + n_8$		N_{28}		$N_{49} + d$
		$a_1 + N_{19}$		N_{39}	
9	$a_0 + 9a_1 + n_9$		N_{29}		
		$a_1 + N_{1,10}$			
10	$a_0 + 10a_1 + n_{10}$				

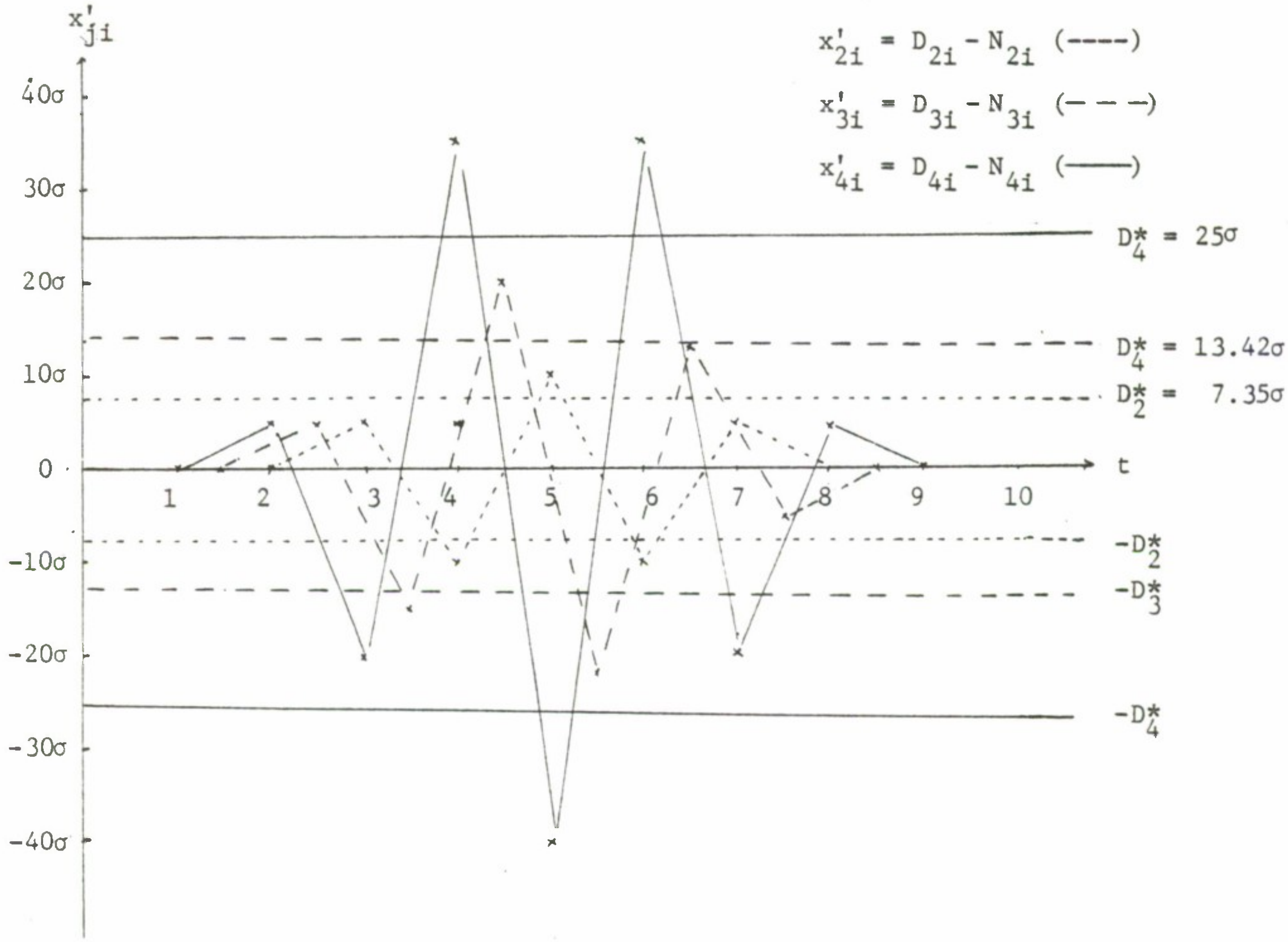


FIGURE 3.4

Disturbance vs Threshold

Two Disturbances Separated by One Point

$d = 5\sigma$

This treatment would introduce the disturbance d in the new value for x_5 and hence to three adjacent equal disturbances. The latter situations presented in Table 3.5 and Figure 3.5. Note, first, that removal of an observation and replacement of the missing point should be followed by recalculation of the ordered differences affected and, second, that the magnitudes of the contributions of the disturbances to the ordered differences are substantially reduced from the contributions in either the isolated disturbance situation or the separated disturbances situation. In this modified situation the reduced thresholds presented in the next section will improve the capability of indicating the presence of the two separated disturbances. A threshold crossing by any of the D_{4i}'s with $i = 3, 4, 5, 6$ in the modified results should serve as an indicator that disturbances may be present in x_4 and x_6 rather than in x_5.

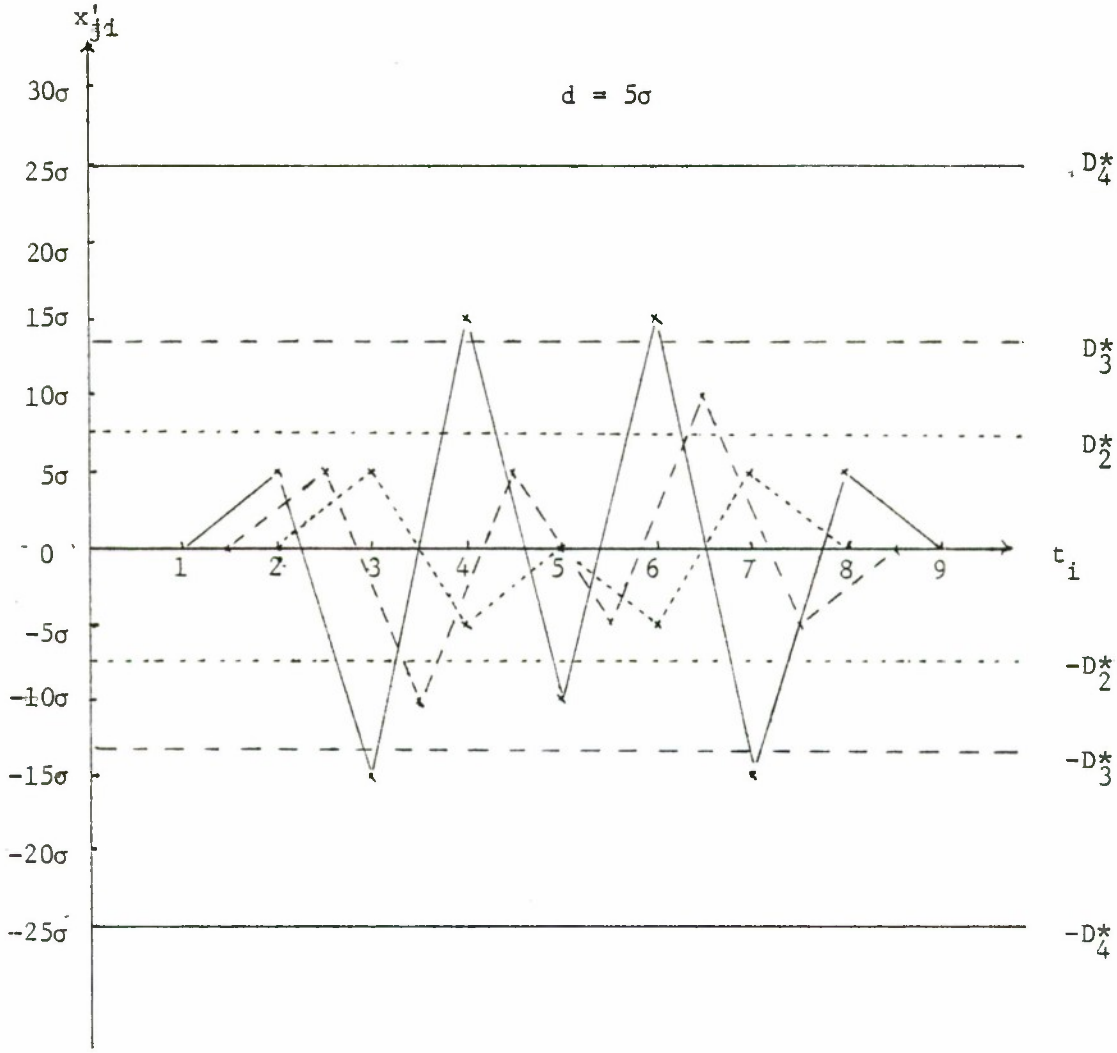
In addition to the occurrence of three adjacent and equal disturbances in the treatment of two such disturbances by replacing missing points, it is possible that this situation can occur due to the persistence of the perturbation causing the disturbances. The lower disturbance contributions to the ordered differences could readily fail to produce a threshold crossing as could the situation with two adjacent equal disturbances whereas the situation with an isolated disturbance

TABLE 3.5

SUCCESSIVE DIFFERENCES

Linear Case: Three Adjacent Equal Disturbances

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a + n_2$		n_{22}		$n_{42} + d$
		$a_1 + n_{13}$		$n_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$n_{23} + d$		$n_{43} - 3d$
		$a_1 + n_{14} + d$		$n_{34} - 2d$	
4	$a_0 + 4a_1 + n_4 + d$		$n_{24} - d$		$n_{44} + 3d$
		$a_1 + n_{15}$		$n_{35} + d$	
5	$a_0 + 5a_1 + n_5 + d$		n_{25}		$n_{45} - 2d$
		$a_1 + n_{16}$		$n_{36} - d$	
6	$a_0 + 6a_1 + n_6 + d$		$n_{26} - d$		$n_{46} + 3d$
		$a_1 + n_{17} - d$		$n_{37} + 2d$	
7	$a_0 + 7a_1 + n_7$		$n_{27} + d$		$n_{47} - 3d$
		$a_1 + n_{18}$		$n_{38} - d$	
8	$a_1 + 8a_1 + n_8$		n_{28}		$n_{48} + d$
		$a_1 + n_{19}$		n_{39}	
9	$a_0 + 9a_1 + n_9$		n_{29}		
		$a_1 + n_{1,10}$			
10	$a_0 + 10a_1 + n_{10}$				



x'_{ji}
30σ
25σ
20σ
15σ
10σ
5σ
0
-5σ
-10σ
-15σ
-20σ
-25σ
$d = 5\sigma$
D_4^*
D_3^*
D_2^*
$-D_2^*$
$-D_3^*$
$-D_4^*$
1
2
3
4
5
6
7
8
9
t_i

FIGURE 3.5

of the same magnitude would yield a threshold crossing. These situations with more than one adjacent, equal disturbances may require greater consideration of the signatures identifying them. (See Figures 3.2 and 3.5.) Such modifications are not examined further in this report.

For the present, it will be assumed that successive differences will be incorporated in a data smoothing algorithm for the two purposes discussed in the introduction (Section I), namely, identifying outliers and indicating appropriate order polynomials for fitting the data. There are two ways that sequential differences can be used in identifying outliers. One is as a preliminary screening to remove some of the more obvious outliers to be followed by a reexamination for outliers in the curve fitting portion of the data smoothing algorithm as presently incorporated in the general track smoothing program MASM3DRJ. The other approach would require sequential differences to provide the only means of identifying outliers. As indicated by the comparatively simple situations considered here, this would require considerably more modal development and become a considerably large portion of a data smoothing program. For the purposes of this report, the first approach will be considered appropriate.

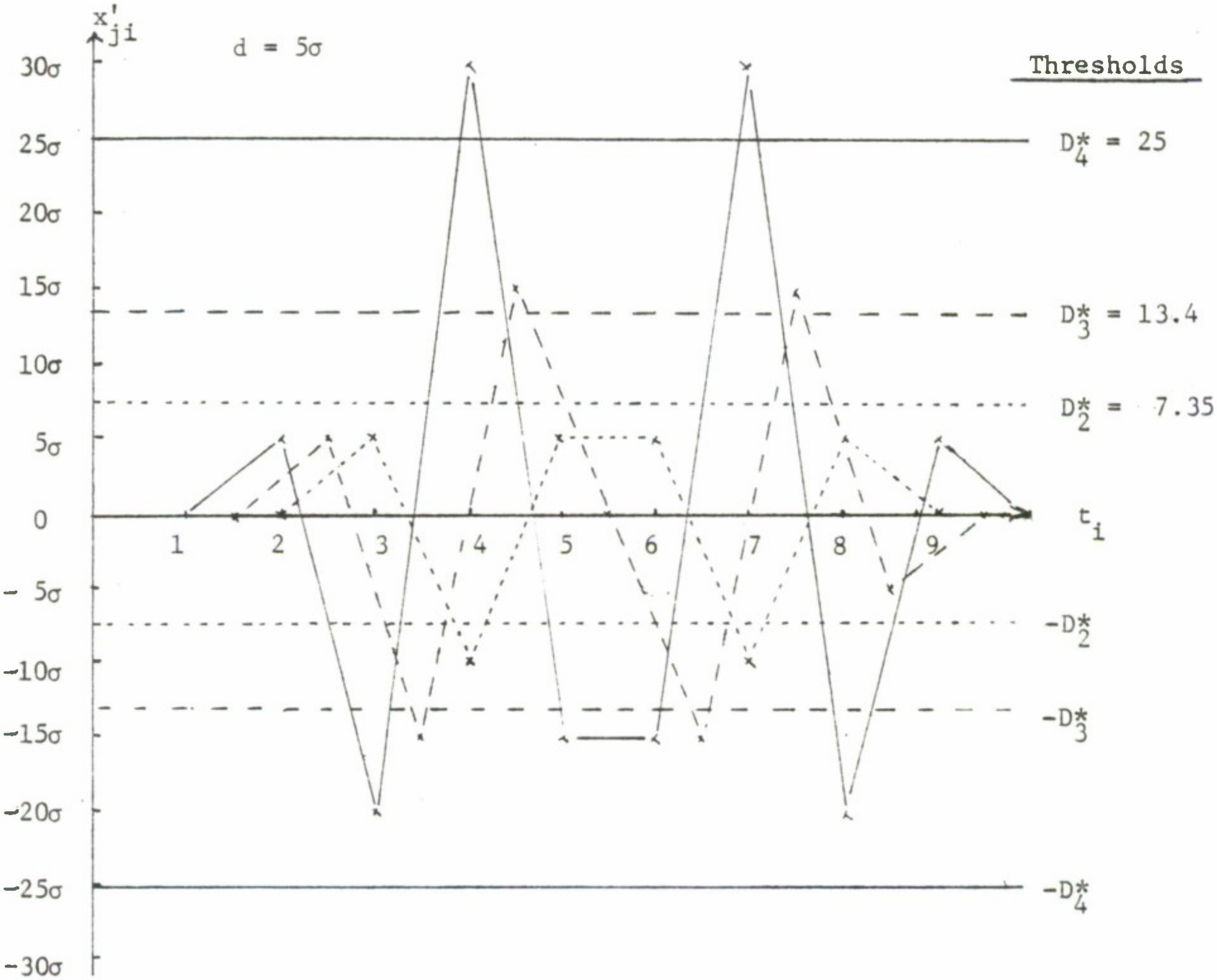
A situation with two equal disturbances separated by two unperturbed observations is presented in Table 3.6 and Figure 3.6. It should be observed that when disturbances are separated by as few as two points they can be considered essentially as isolated disturbances. (See Table 3.1 and Figure 3.1.)

TABLE 3.6

SUCCESSIVE DIFFERENCES

Linear Case: Two Equal Disturbances Separated by Two Points

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		$n_{42} + d$
		$a_1 + n_{13}$		$n_{33} + d$	
3	$a_0 + 3a_1 + n_3$		$n_{23} + d$		$n_{43} - 4d$
		$a_1 + n_{14} + d$		$n_{34} - 3d$	
4	$a_0 + 4a_1 + n_4 + d$		$n_{24} - 2d$		$n_{44} + 6d$
		$a_1 + n_{15} - d$		$n_{35} + 3d$	
5	$a_0 + 5a_1 + n_5$		$n_{25} + d$		$n_{45} - 3d$
		$a_1 + n_{16}$		n_{36}	
6	$a_0 + 6a_1 + n_2$		$n_{26} + d$		$n_{46} - 3d$
		$a_1 + n_{17} + d$		$n_{37} - 3d$	
7	$a_0 + 7a_1 + n_7 + d$		$n_{27} - 2d$		$n_{47} + 6d$
		$a_1 + n_{18} - d$		$n_{38} + 3d$	
8	$a_0 + 8a_1 + n_8$		$n_{28} + d$		$n_{48} - 4d$
		$a_1 + n_{19}$		$n_{39} - d$	
9	$a_0 + 9a_1 + n_9$		n_{29}		$n_{49} + d$
		$a_1 + n_{1,10}$		$n_{3,10}$	
10	$a_0 + 10a_1 + n_{10}$		$n_{2,10}$		
		$a_1 + n_{1,11}$			
11	$a_0 + 11a_1 + n_{11}$				



x'_{ji}
d = 5σ
Thresholds
30σ
25σ
20σ
15σ
10σ
5σ
0
- 5σ
-10σ
-15σ
-20σ
-25σ
-30σ
1
2
3
4
5
6
7
8
9
t_i
$D_4^* = 25$
$D_3^* = 13.4$
$D_2^* = 7.35$
$-D_2^*$
$-D_3^*$
$-D_4^*$

FIGURE 3.6

There are other types of perturbations that could, and possibly should, be considered for potential identification by successive differences. Only one of these will be examined here. This is the situation in which the torpedo changes from a linear path at t_r to a different linear path at t_{r+1}. This situation is presented in Table 3.7 and Figure 3.7. As can be seen by comparison of Table 3.7 with Table 3.1, it is possible that a path change at $t = r$ could lead to the identification of x_r as containing a disturbance d depending on the magnitude of Δ_1 and d. The resemblance of the signature (graph) of D_{4i} in the two situations could be even more striking for a value of d such that $D_{4,2}$ of Table 3.1 (corresponding to $D_{4,r-2}$ of Table 3.7) were small enough to be submerged in noise and $\Delta_1 \doteq 3d$. That a path change could conceivably cause a threshold crossing of D_4^* by D_{4r} can be seen in the case of a 90° change from $\theta = 0$ to $\theta' = 90^\circ$ (or, vice versa) where $|\Delta_1| = |\vec{V}| \doteq 90$. The situation is even worse for a 90° change from $\theta = 45^\circ$ to $\theta' = 135^\circ$ with $|\Delta_1| \doteq 1.4(90) = 126$.

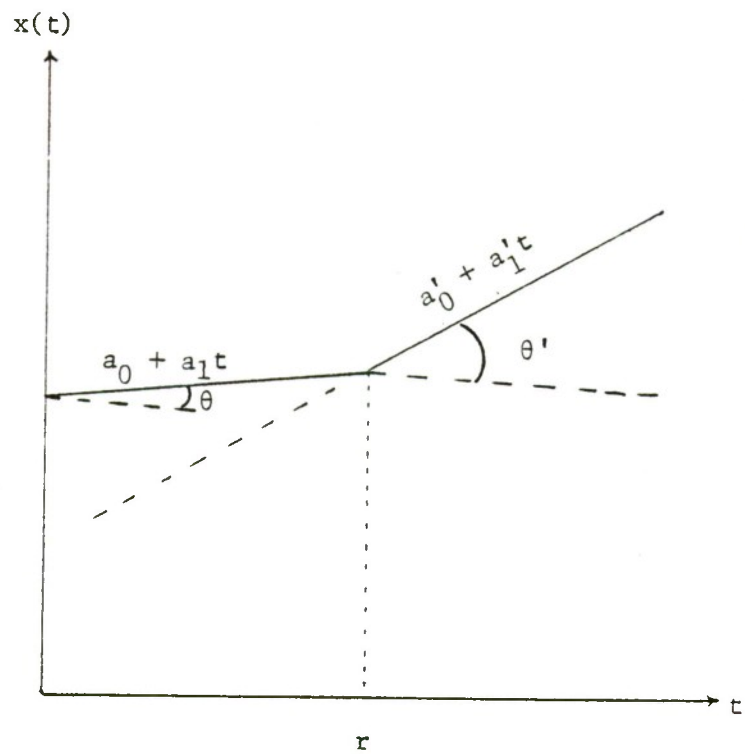
Possible methods of identifying path changes to prevent mis-identification as outliers include reconsideration of labeled outliers after fitting curves to the data and provision from an external source such as control information. The first method requires greater complication of the data smoothing program involving cycling and hence negates the intent of a simple screening program for outliers. The second requires

TABLE 3.7

Linear Case: Path Change at $t_i = r$

$$x(t_i) = \begin{cases} a_0 + a_1 t_i + n_i, & t_i \leq r; \\ a_0' + a_1' t_i + n_i, & t_i > r; \end{cases} \quad ; \quad x(r) = \begin{cases} a_0 + r a_1 + n_r \\ a_0' + r a_1' + n_r \end{cases} \quad ; \quad s_1 = a_1' - a_1$$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
r-4	$a_0 + (r-4)a_1 + n_{r-4}$				
		$a_1 + n_{1,r-3}$			
r-3	$a_0 + (r-3)a_1 + n_{r-3}$		$n_{2,r-3}$		
		$a_1 + n_{1,r-2}$		$n_{3,r-2}$	
r-2	$a_0 + (r-2)a_1 + n_{r-2}$		$n_{2,r-2}$		$n_{4,r-2}$
		$a_1 + n_{1,r-1}$		$n_{3,r-1}$	
r-1	$a_0 + (r-1)a_1 + n_{r-1}$		$n_{2,r-1}$		$n_{4,r-1} + \Delta_1$
		$a_1 + n_{1,r}$		$n_{3,r} + \Delta_1$	
r	$x(r)$		$n_{2,r} + \Delta_1$		$n_{4,r} - 2\Delta_1$
		$a_1' + n_{1,r+1}$		$n_{3,r+1} - \Delta_1$	
r+1	$a_0' + (r+1)a_1' + n_{r+1}$		$n_{2,r+1}$		$n_{4,r+1} + \Delta_1$
		$a_1' + n_{1,r+2}$		$n_{3,r+3}$	
r+2	$a_0' + (r+2)a_1' + n_{r+2}$		$n_{2,r+2}$		$n_{4,r+2}$
		$a_1' + n_{1,r+3}$		$n_{3,r+3}$	
r+3	$a_0' + (r+3)a_1' + n_{r+3}$		$n_{2,r+3}$		
		$a_1' + n_{1,r+2}$			
r+4	$a_0' + (r+4)a_1' + n_{r+4}$				



$$a_1 = |\vec{V}| \cos \theta$$

$$a_1' = |\vec{V}| \cos \theta'$$

$$\Delta_1 = |\vec{V}| (\cos \theta' - \cos \theta)$$

FIGURE 3.7

input information from another source and is also undesirable but to a lesser extent. An alternative treatment is to accept such identification of point of path change as providing an outlier to be removed from the data. The consequences of this treatment will be examined in a subsequent report on curve fitting and appears, at least for the present, to be a reasonable way of handling the situation.

There is still another kind of perturbation which can, and has been observed to occur. This is a change in the noise component and represented by a change in the value of the standard deviation σ. Such changes may be a result of changes in the environment or of the data gathering system. Evidence of such changes in the value of σ should be accommodated by corresponding changes in the threshold levels.

F. Missing Points

The occurrence of missing observations in a sequence of observations needs some consideration. A missing observation can be present in the data input or occur as a result of deletion of an outlier. Note that, in the latter case, recalculation of successive differences will be required in the vicinity of the deleted observation.

As the simplest procedure for replacing missing points, the currently used procedure of averaging over the adjacent points will be used here. (This also will be re-examined when curve-fitting is discussed.) Thus, when x_r is missing it it will be replaced by

$$x_r' = \frac{1}{2}\,(x_{r-1} + x_{r+1})$$

and when adjacent values x_r and x_{r+1} are missing they will be replaced by

$$x_r' = x_{r-1} + \frac{1}{3}\,(x_{r+2} - x_{r-1}) = \frac{1}{3}\,(2x_{r-1} + x_{r+2})$$

$$x_{r+1}' = x_{r-1} + \frac{2}{3}\,(x_{r+2} = x_{r-1}) = \frac{1}{3}\,(x_{r-1} + 2x_{r+2}) \ .$$

The general formula for k successive missing points is

$$x_{r+j}' = x_{r-1} + \frac{j+1}{k+1}\,(x_{r+k} - x_{r-1}) \qquad \text{for} \quad j = 0,\ldots,k-1 \ .$$

There is a serious question, however, if an analysis of successive differences is improved by replacement of more than two successive missing values. It would appear more reasonable, at least on examination of the fourth order successive differences which involve only sequences of five observations, to restart calculation of successive differences at the first observation after a sequence of more than two missing observations.

The situation involving a missing point with linear polynomial and noise components only is presented in Tables 4.1 and the accompanying definitions for the modified noise components with their variances. Reduced thresholds could be used as indicated in Table 4.2 and Figure 4.1. These reduced

TABLE 4.1

Linear Case: Missing Point (x_4) Averaged

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		n^*_{42}
		$a_1 + n_{13}$		n^*_{33}	
3	$a_0 + 3a_1 + n_3$		n^*_{23}		n^*_{43}
		$a_1 + n^*_{14}$		n^*_{34}	
4	$a_0 + 4a_1 + \frac{n_3 + n_5}{2}$		$n^*_{24} = 0$		n^*_{44}
		$a_1 + n^*_{15}$		n^*_{35}	
5	$a_0 + 5a_1 + n_5$		n^*_{25}		n^*_{45}
		$a_1 + n_{16}$		n^*_{36}	
6	$a_0 + 6a_1 + n_6$		n_{26}		n^*_{46}
		$a_1 + n_{17}$		n_{37}	
7	$a_0 + 7a_1 + n_7$		n_{27}		
		$a_1 + n_{18}$			
8	$a_8 + 8a_1 + n_1$				

TABLE 4.1 Continued

$$x_4^* = \frac{x_3 + x_5}{2} = a_0 + 4a_1 + \frac{n_3 + n_5}{2}, \qquad \sigma^2_{x_4^*} = \frac{\sigma^2}{2}$$

$$n_{15}^* = n_{14}^* = \frac{n_5 - n_3}{2}, \qquad \sigma^2_{n_{14}^*} = \frac{\sigma^2}{2}$$

$$n_{23}^* = \frac{1}{2}(n_5 - 3n_3 + 2n_2), \qquad \sigma^2_{n_{23}^*} = \frac{7}{2}\sigma^2$$

$$n_{24}^* = 0$$

$$n_{25}^* = \frac{1}{2}(2n_6 - 3n_5 + n_3), \qquad \sigma^2_{n_{25}^*} = \frac{7}{2}\sigma^2$$

$$n_{33}^* = \frac{1}{2}(n_5 - 5n_3 + 6n_2 + 2n_1), \qquad \sigma^2_{N_{33}^*} = \frac{33}{2}\sigma^2$$

$$n_{34}^* = -n_{23}^*,$$

$$n_{35}^* = n_{25}^*$$

$$n_{36}^* = \frac{1}{2}(2n_7 - 6n_6 + 5n_5 - n_3), \qquad \sigma^2_{n_{36}^*} = \frac{33}{2}\sigma^2$$

$$n_{42}^* = \frac{1}{2}(n_5 - 7n_3 + 12n_2 - 8n_1 + n_0), \qquad \sigma^2_{n_{42}^*} = \frac{131}{2}\sigma^2$$

$$n_{43}^* = -n_5 + 4n_3 - 4n_2 + n_1, \qquad \sigma^2_{n_{42}^*} = 34\sigma^2$$

$$n_{44}^* = n_6 - 2n_5 - n_3 + n_2, \qquad \sigma^2_{n_{44}^*} = 7\sigma^2$$

$$n_{45}^* = n_7 - 4n_6 + 4n_5 - n_3, \qquad \sigma^2_{n_{45}^*} = 34\sigma^2$$

$$n_{46}^* = \frac{1}{2}(2n_8 - 8n_7 + 12n_6 - 7n_5 + n_3), \qquad \sigma^2_{n_{46}^*} = \frac{131}{2}\sigma^2$$

$$\sigma^2_{n_{ji}^*} < \sigma^2_{n_{ji}} \quad \text{for all } j, i.$$

TABLE 4.2

Linear Case: Detection Thresholds for Missing Point Datum at r

$D^*_{1i} = 3\sigma_{n_{ji}}$, Table Values for $3\sigma_{ji}/\sigma$

t_i	D_n/σ	$(D^*_{1i}-a)/\sigma$	D^*_{2i}/σ	D^*_{3i}/σ	D^*_{4i}/σ
r-4	3				
		4.24			
r-3	3		7.35		25.1
		4.24		13.4	
r-2	3		7.35		24.3
		4.24		12.2	
r-1	3		5.51		17.5
		2.10		5.51	
r	2.1		0		8.0
		2.10		5.51	
r+1	3		5.51		17.5
		4.24		12.2	
r+2	3		7.35		24.3
		4.24		13.4	
r+3	3		7.35		25.1
		4.24			
r+4	3				

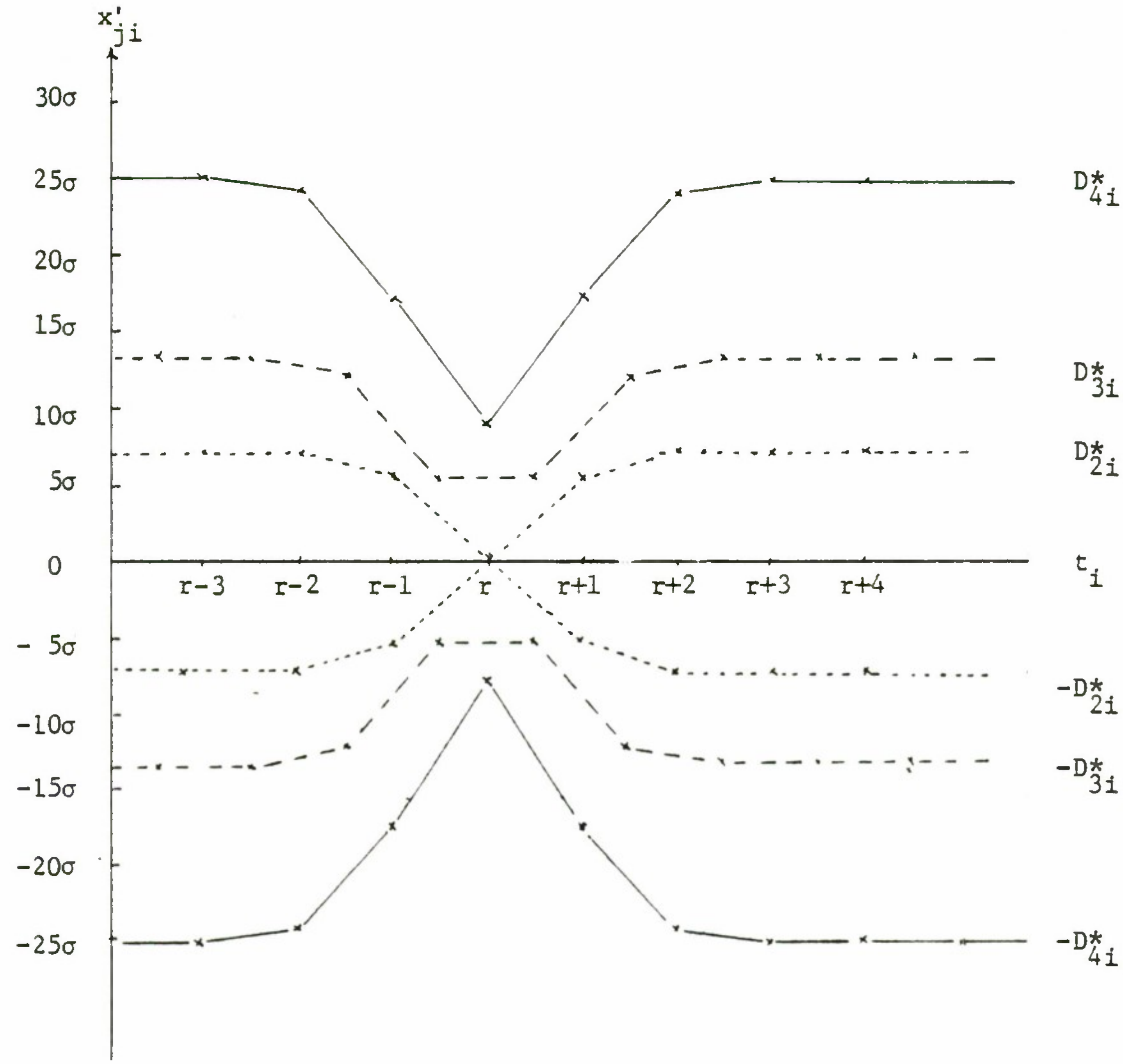


FIGURE 4.1

Thresholds in Vicinity of a Missing Point thresholds could be useful in identifying situations involving equal disturbances separated by one observation where that observation is labeled as an outlier and replaced by the average of the two observations with disturbances. Recalculation of the fourth order differences produces the disturbance components given in the last column of Table 3.5 which are shown with the modified thresholds in Figure 4.2. (This situation is the same as for two disturbances separated by a missing point.) Persistence of a threshold crossing at t_r after deletion and replacement of the observation x_r can be an indication that disturbances may be present in x_{r-1} and x_{r+1} instead of, or in addition to, a disturbance in x_r.

Some additional work is required here to assist in developing that portion of the data smoothing program dealing with successive differences. It is fairly clear that the existence of a threshold crossing requires more effort to determine whether it indicates an isolated outlier or a more complicated situation. A situation with two adjacent missing observations and no disturbances is displayed in Table 4.3 accompanied by the expressions for the noise components in terms of the observational noise. The variances for the noise components presented there provide the basis for the thresholds shown in Table 4.4. The thresholds for the isolated missing point situation are also shown in Table 4.4. Note that the

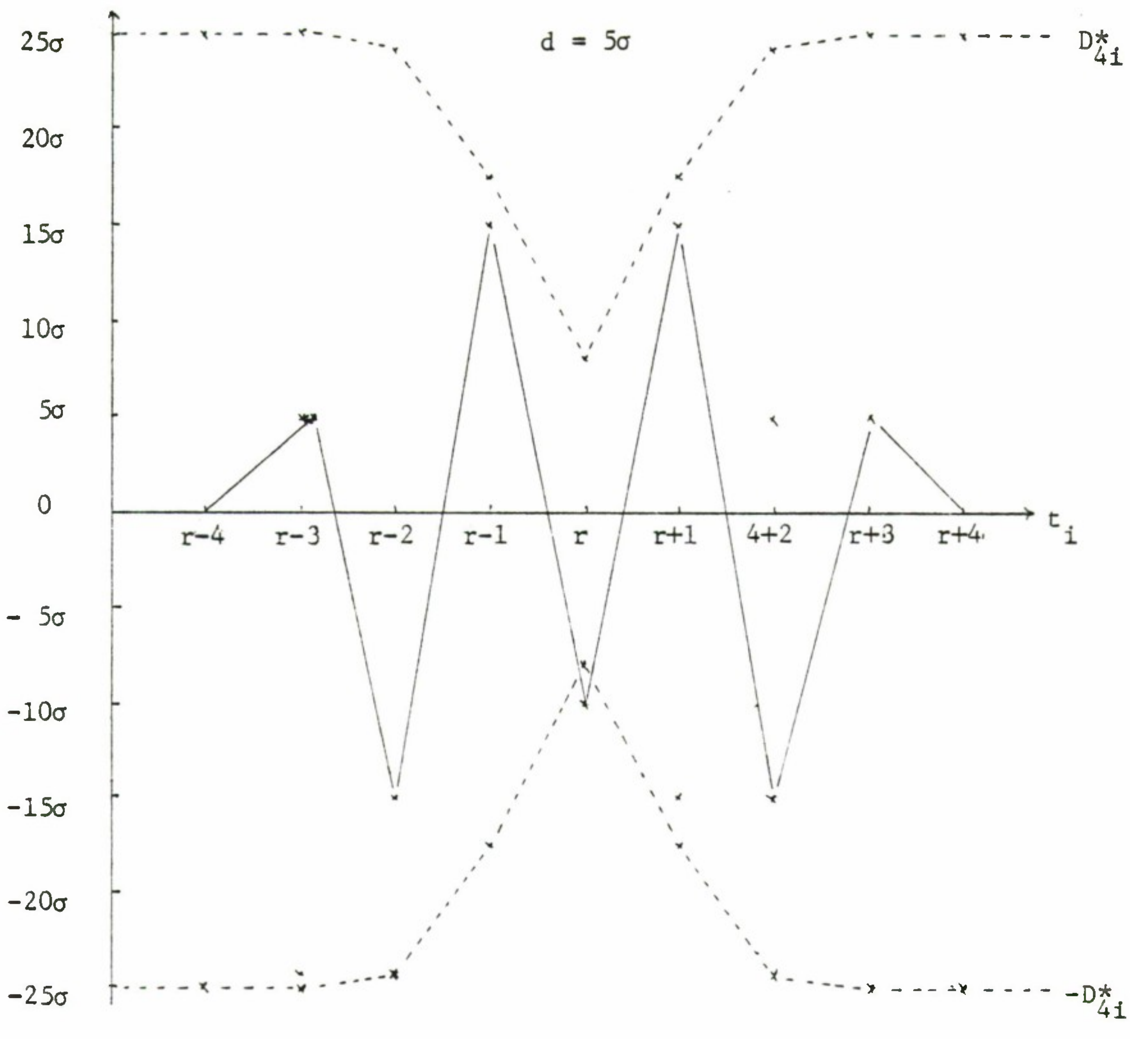


FIGURE 4.2

Two Disturbances Separated by a Missing Point Averaged

TABLE 4.3

Linear Case: Adjacent Missing Points Averaged

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		n^*_{42}
		$a_1 + n_{13}$		n^*_{33}	
3	$a_0 + 3a_1 + n_3$		n^*_{23}		n^*_{43}
		$a_1 + n^*_{14}$		n^*_{34}	
4	x^*_4		$n^*_{24} = 0$		n^*_{44}
		$a_1 + n^*_{15}$		$n^*_{35} = 0$	
5	x^*_5		$n^*_{25} = 0$		n^*_{45}
		$a_1 + n^*_{16}$		n^*_{36}	
6	$a_0 + 6a_1 + n_6$		n^*_{26}		n^*_{46}
		$a_1 + n_{17}$		n^*_{37}	
7	$a_0 + 7a_1 + n_7$		n_{27}		n^*_{47}
		$a_1 + n_{18}$		n_{38}	
8	$a_0 + 8a_1 + n_8$		n_{28}		
		$a_1 + n_{19}$			
9	$a_0 + 9a_1 + n_9$				

TABLE 4.3 Continued

$$D_4^* = x_3 + \frac{1}{3}(x_6 - x_3) = a_0 + 4a_1 + n_4^*, \quad n_4^* = \frac{1}{3}n_6 + \frac{2}{3}n_3, \quad \sigma^2_{n_4^*} = \frac{5}{9}\sigma^2$$

$$D_5^* = x_3 + \frac{2}{3}(x_6 - x_3) = a_0 + 5a_1 + n_5^*, \quad n_5^* = \frac{2}{3}n_6 + \frac{1}{3}n_3, \quad \sigma^2_{n_5^*} = \frac{5}{9}\sigma^2$$

$$n_{14}^* = \frac{1}{3}(n_6 - n_3), \qquad n_{26}^* = \frac{1}{3}(3n_7 = 4n_6) + n_3), \quad \sigma^2_{n_{23}^*} = \sigma^2_{n_{26}^*} = 2\frac{8}{9}\sigma^2$$

$$n_{23}^* = \frac{1}{3}(n_6 - 4n_3 + 3n_2), \quad n_{26}^* = \frac{1}{3}(3n_7 - 4n_6 + n_3), \quad \sigma^2_{n_{23}^*} = \sigma^2_{n_{26}^*} = 2\frac{8}{9}\sigma^2$$

$$n_{24}^* = n_{25}^* = 0$$

$$n_{33}^* = \frac{1}{3}(n_6 - 7n_3 + 9n_2 - 3n_1), \quad n_{37}^* = \frac{1}{3}(3n_8 - 9n_7 + 7n_6 - n_3), \quad \sigma^2_{n_{33}^*} = \sigma^2_{n_{37}^*} = 15\frac{5}{9}\sigma^2$$

$$n_{34}^* = -n_{23}^*, \quad n_{35}^* = 0, \quad n_{36}^* = n_{26}^*$$

$$\left.\begin{array}{l} n_{42}^* = \frac{1}{3}(n_6 - 10n_3 + 18n_2 - 12n_2 + 3n_0) \\[1em] n_{47}^* = \frac{1}{3}(3n_9 - 12n_8 + 18n_7 - 10n_6 + n_3) \end{array}\right\} \quad \sigma^2_{n_{42}^*} = \sigma^2_{n_{47}^*} = 64\frac{2}{9}\sigma^2$$

$$\left.\begin{array}{l} n_{43}^* = \frac{1}{3}(-2n_6 + 11n_3 - 12n_2 + 3n_1) \\[1em] n_{46}^* = \frac{1}{3}(3n_8 - 12n_7 + 1/n_6 - 2n_3) \end{array}\right\} \quad \sigma^2_{n_{43}^*} = \sigma^2_{n_{46}^*} = 30\frac{8}{9}\sigma^2$$

$$n_{44}^* = -n_{34}^*, \quad n_{45}^* = n_{36}^*$$

TABLE 4.4

THRESHOLDS FOR NOISE IN ONE AND TWO MISSING POINT SITUATIONS

k such that $D^*_{ji} = 3\sigma_{n_{ji}} = k\sigma$

t_i \ # Pts miss / j	1					2				
	0	1	2	3	4	0	1	2	3	4
r-4										
r-3					25.1					25.1
				13.4					13.4	
r-2			7.35		24.3			7.35		24.0
		4.24		12.2			4.24		11.8	
r-1	3		5.61		17.5	3		5.1		16.7
		2.10		5.61			1.41		5.1	
r	2.10		0		8.0	2.24		0		5.1
		2.10		5.61			1.41		0	
r+1	3		5.61		17.5	2.24		0		5.1
		4.24		12.2			1.41		5.1	
r+2			7.35		24.3	3		5.1		16.7
				13.4			4.24		11.8	
r+3					25.1			7.35		24.0
r+4									13.4	
r+5										25.1

thresholds in the two missing points situation are smaller than the corresponding ones in a situation with a single missing point.

A situation in which a disturbance occurs in an observation adjacent to a missing point is presented in Table 4.5 (It is suspected that in situations involving one or more missing points, could also involve disturbances immediately preceding of following a missing point due to deteriorization of physical conditions.) The disturbance components are shown in relationship to the common thresholds appropriate when there are no missing points in Figure 4.3 and to the reduced thresholds in Figures 4.4, 4.5 and 4.6. It can be seen that the use of the modified thresholds can increase the potential crossing of thresholds in the vicinity of a missing point substantially.

Examination of the effects of missing points on the ability of successive differences to indicate the presence of disturbances is not complete. For example, situations with disturbances preceding and/or following adjacent missing points have not been examined. Nevertheless, some indications of the consideration of missing points in the use of successive differences to screen 3-D data for outliers can be suggested at this point in the development. Under the guiding principle of keeping the data smoothing program as short and simple as possible, and with the understanding that a further screening for outliers could be included in the curve fitting portion of the program, the following steps appear reasonable:

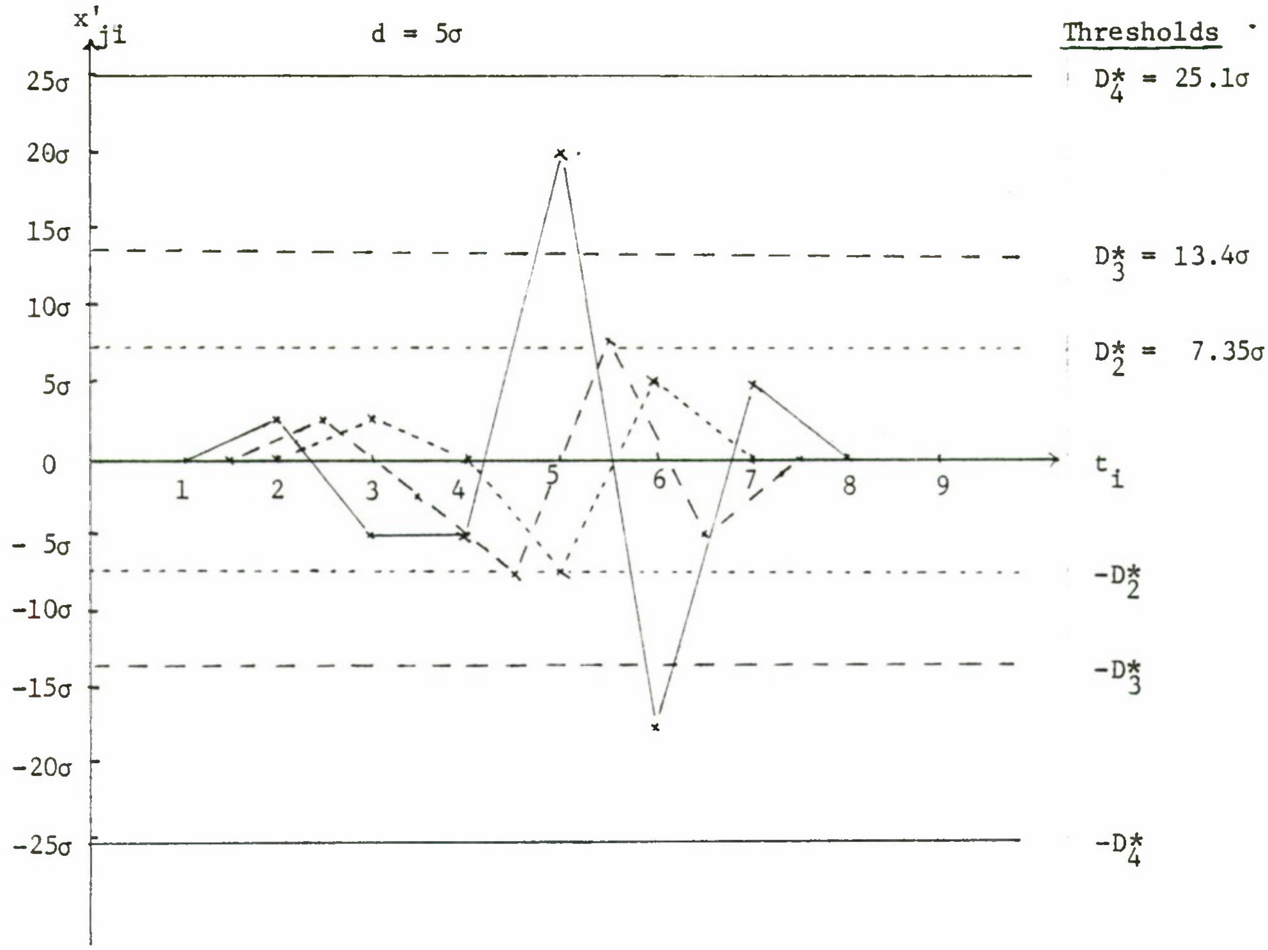
TABLE 4.5

Linear Case: Disturbance Following Missing Point

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
0	$a_0 + n_0$				
		$a_1 + n_{11}$			
1	$a_0 + a_1 + n_1$		n_{21}		
		$a_1 + n_{12}$		n_{32}	
2	$a_0 + 2a_1 + n_2$		n_{22}		$n^*_{42} + \frac{d}{2}$
		$a_1 + n_{13}$		$n^*_{33} + \frac{d}{2}$	
3	$a_0 + 3a_1 + n_3$		$n^*_{23} + \frac{d}{2}$		$n^*_{43} - d$
		$a_1 + n^*_{14} + \frac{d}{2}$		$n^*_{34} = \frac{d}{2}$	
4	x^*_4		$n^*_{24} = 0$		$n^*_{44} - d$
		$a_1 + n^*_{15} + \frac{d}{2}$		$n^*_{35} - \frac{3}{2} d$	
5	$a_0 + 5a_1 + n_5 + d$		$n^*_{25} - \frac{3}{2} d$		$n^*_{45} + 4d$
		$a_1 + n_{16} - d$		$n^*_{36} + \frac{5}{2} d$	
6	$a_0 + 6a_1 + n_6$		$n_{26} + d$		$n^*_{46} + \frac{7}{2} d$
		$a_1 + n_{17}$		$n_{37} - d$	
7	$a_0 + 7a_1 + n_7$		n_{27}		$n_{47} + d$
		$a_1 + n_{18}$		n_{38}	
8	$a_0 + 8a_1 + n_8$		n_{28}		
		$a_1 + n_{19}$			
9	$a_0 + 9a_1 + n_9$				

$x^*_4 = \frac{1}{2}(x_3 + x_5) = a_0 + 4a_1 + n^*_4 + d/2, \quad n^*_4 = \frac{1}{2}(n_3 + n_5)$

(For n^*_{ji}'s see Missing Point Table, Table 4.1.)



x'ji
d = 5σ
Thresholds
D*4 = 25.1σ
D*3 = 13.4σ
D*2 = 7.35σ
-D*2
-D*3
-D*4
25σ
20σ
15σ
10σ
5σ
0
- 5σ
-10σ
-15σ
-20σ
-25σ
1
2
3
4
5
6
7
8
9
ti

FIGURE 4.3

Linear Case: Second Order Differences vs Thresholds
Disturbance Following Missing Point

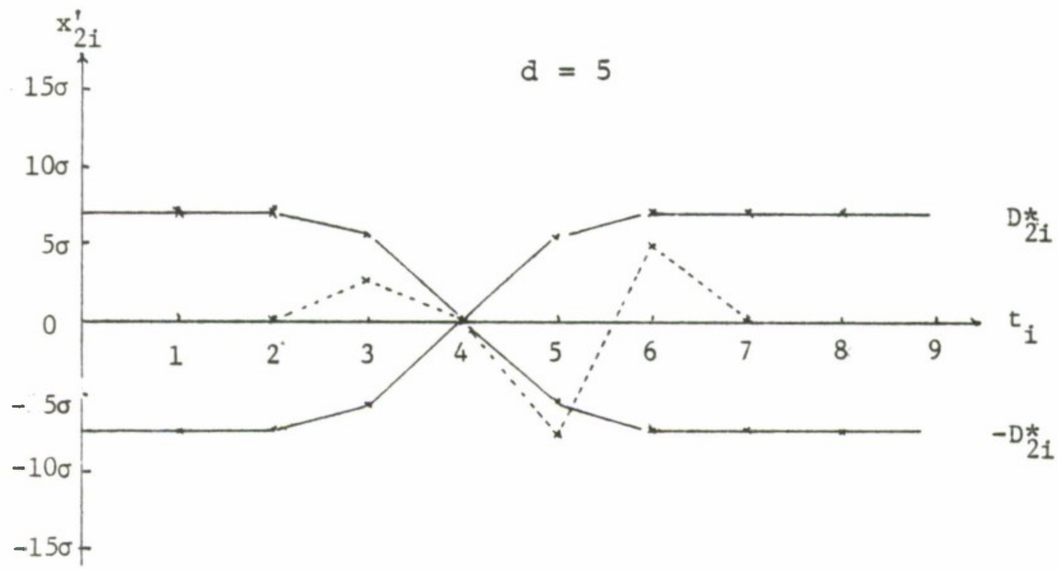


FIGURE 4.4

Linear Case: Third Order Differences vs Thresholds
Disturbance Following Missing Point

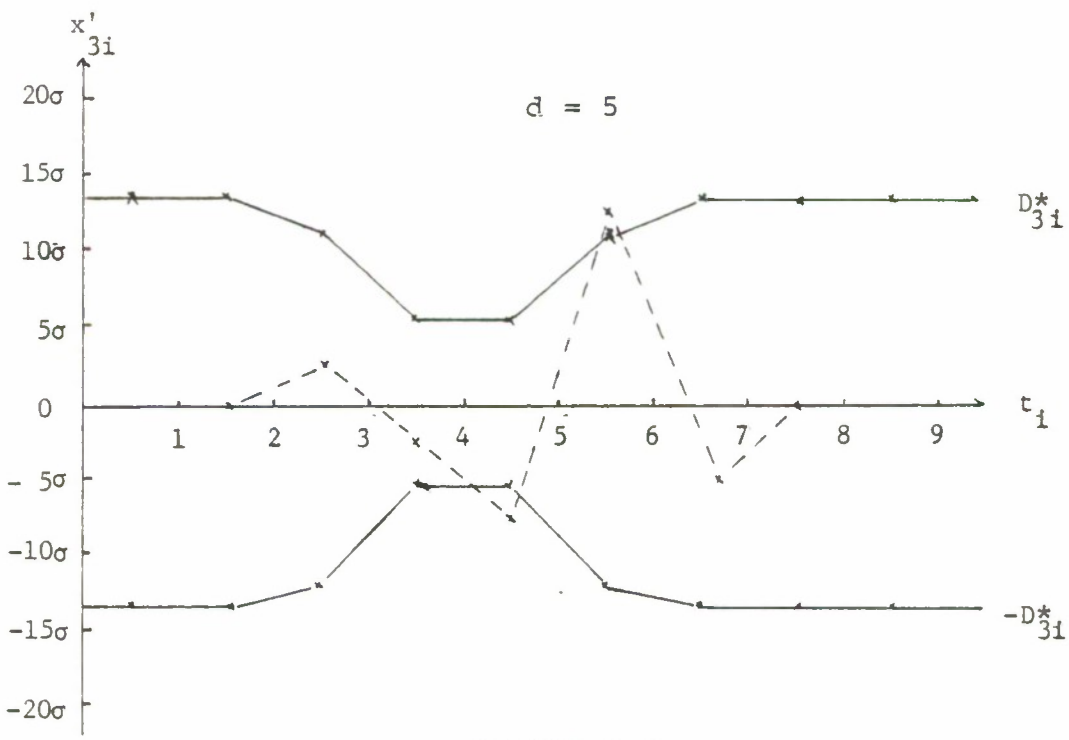


FIGURE 4.5

Linear Case: Fourth Order Differences vs Thresholds
Disturbance Following Missing Point

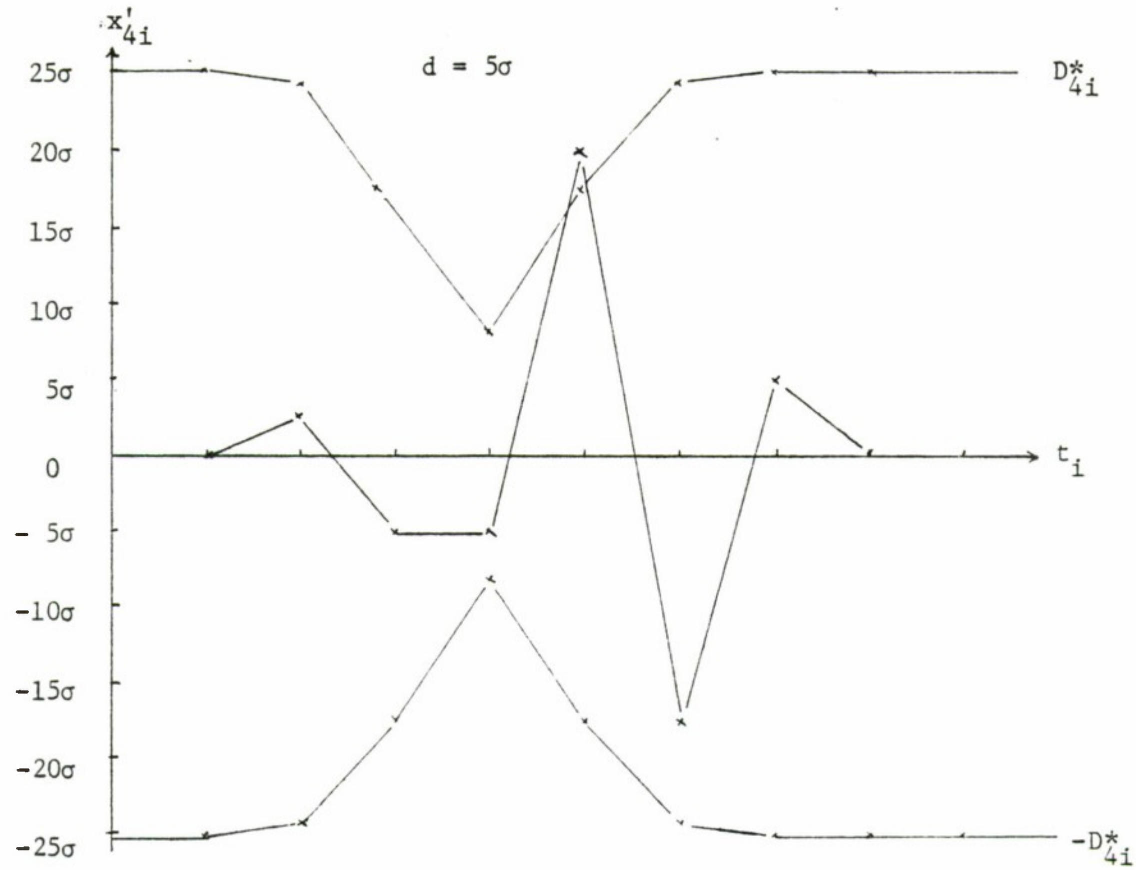


FIGURE 4.6

(1) Supply missing points using the averaging method.

(2) Screen for outliers using the fourth order differences D_{4i} and the common threshold D_4^*.

(3) Replace any outliers found by the averaging method.

(4) Screen for outliers in the vicinity of any values replaced in Step 3 (not those in Step 1) using the reduced thresholds D_{4i}^* for the D_{4i}'s.

(5) Any outliers found in Step 4 should be referred for manual examination, at least until further development can provide satisfactory provisions for inclusion in the smoothing program.

G. Noise Variance

In Section 2.D, it was assumed that the noise components of the data were normally and independently distributed with zero means and common variance σ^2. This variance, or more specifically the standard deviation σ, must be known before the thresholds discussed in Sections 2.D, E, and F can be specified. Selection of an appropriate value for σ requires more detailed examination. Three potential sources of values for σ will be considered here.

In Reference 1, which incidentally used path segments from the same set of data to be used in this study, sample standard deviations of magnitudes S_x = 2 or 3 were calculated for some path segments. Sample standard deviations provide the primary sources of information on the value of σ and hence are of considerable interest in setting threshold values. They can be, unfortunately, contaminated by the polynomial components in the observations as was demonstrated in the reference. Nevertheless, a value of the order of $\sigma = 3$ or $\sigma = 4$ is an approximation which could be used in setting thresholds for screening for outliers. Experience with larger samples including other runs will provide a more reasonable basis for estimating σ.

It is to be expected that there will be spatial and temporal variations in σ. Spatial variables can be present because of the geometry of the vehicle-sensor orientation. Data from which the value of σ and its spatial variations should be available from previous and continuing calibration data collected on the position location system. Information on temporal variations should be available from the same source and should also be monitored during the collection of any data for which data smoothing is to be performed. It should also be expected that there will be interaction between spatial and temporal variation in σ, i.e., that the temporal variation can be different for different locations on the path of the vehicle being tracked. This would imply that the thresholds to be used for indicating outliers may, and probably should, be changed depending on the location and time of the data to be smoothed.

The third potential source for information on σ is the data to be smoothed. A single estimate S_x for $\sigma_x = \sigma$ may be calculated from the complete set of data or estimates may be calculated for segments of the data. These can be expected to be contaminated by both the polynomial and perturbation components in the data. Reduction in the polynomial component contribution could be obtained by using successive differences as the source of the estimates. Thus, for example, the sample variance of the fourth order differences

$$S_{D_4}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (D_{4i} - \bar{D}_4)^2$$

where

$$\bar{D}_4 = \frac{1}{n} \sum_{i=1}^{n} D_{4i}$$

could be used as an estimate of

$$\sigma_{D_4}^2 = 70\sigma^2$$

leading to the estimate

$$\hat{\sigma} = \sqrt{\frac{1}{70} S_{D_4}}$$

This should have little or no contamination from the poly nomial components of the observations. If the outliers are reasonably rare, the perturbation contributions should also be small and the resulting estimate could be a reasonable alternative. Note that estimates of σ could be obtained for segments of the data and hence could be made to respond to the spatial and temporal variations in σ discussed in the second alternative.

This third method of estimating σ has a direct relationship to the method (Grubb's) incorporated in the currently used program for identifying outliers. In fact, a Grubb's type of screening could be performed with the sample variances of successive differences where an observation is labeled an outlier if its removal provides a substantial reduction in the sample variance. This possibility has not been explored.

H. Algorithm for Identifying Outliers

The following algorithm is suggested for identifying and removal of gross outliers. Two basic principles are considered essential:

(1) The algorithm should be simple and short.

(2) A subsequent and more thorough search for outliers will be incorporated in the data smoothing program concurrent with or following the curve-fitting portion of the program.

The steps of the algorithm are:

1. Calculate values for missing points using the method of averaging.
2. Calculate the fourth order differences D_{4i}.

3. Identify as outliers and remove from the data any x_k for which $|D_{4k}| > 25.1\sigma$. (The suggested value of σ to be used here is a number between 3 and 4.)

4. Replace any x_k identified as an outlier in Step 3 using the averaging method as in Step 1.

5. Recalculate the fourth order differences which involve x_k. (These are $D_{4,k-2}, \cdots, D_{4,k+2}$.)

6. Re-examine the modified fourth order differences of Step 5 for outliers as in Step 3.

7. If additional outliers are found in Step 6, either additional steps must be designed to locate potential outliers in the vicinity of the observation x_k (from Step 3) or the problem must be identified for manual treatment.

I. Identifying Polynomial Components

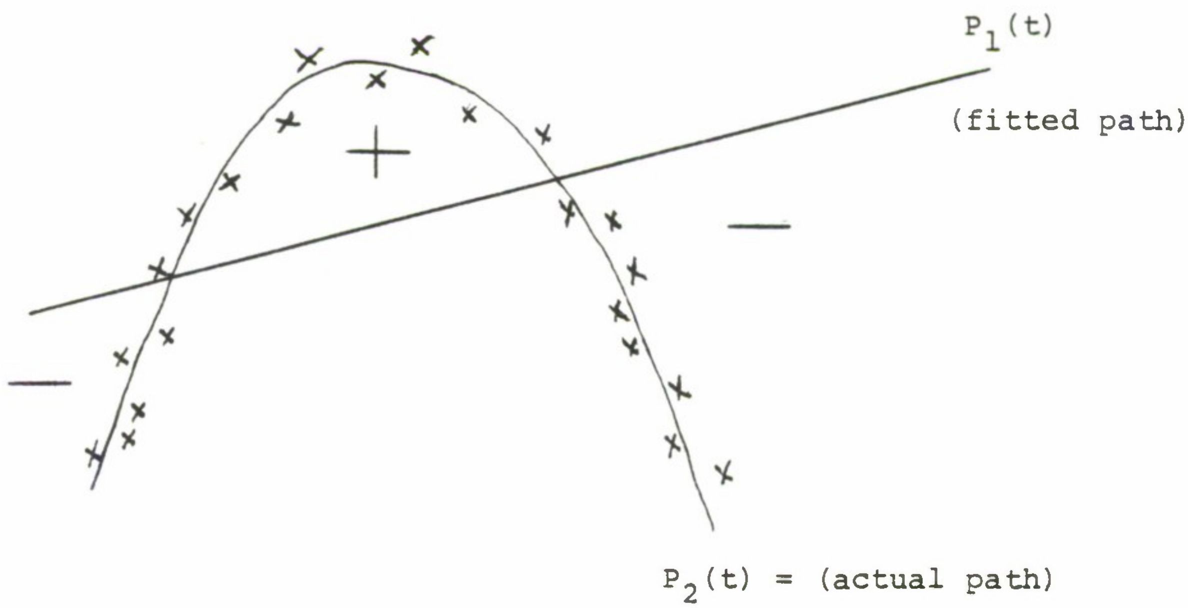
In using successive differences to indicate the appropriate degree of the polynomial component $P(t)$, attention is directed to the sequence of signs of the differences of the same order. The reasoning for this is as follows. In Section 2.D it was established that the noise component n_{ji} of the i^{th} difference of order j is a linear combination of the n_i's (the noise components of the observations). If the N_i's (the random variables of which the n_i's are realizations and hence the noise components of the observations, x_i's) have zero means as assumed in Section 2.A, then the N_{ji}'s will also have zero means. In any sequence of differences of order j, the mean value of the differences $\bar{N}_j = \frac{1}{n} \sum_{i=1}^{N} N_{ji}$ will also have zero mean. In the absence of a polynomial component with a term $a_r t^r$ and without a disturbance component, the r^{th} order difference terms $D_{ri} = n_{ri}$ and hence the mean value

$$\bar{D}_r = \frac{1}{n} \sum_{1}^{n} D_{ri} = \frac{1}{n} \sum_{1}^{n} n_{ri}$$

should be near zero. The occurrence of a sequence of differences of order r having the same sign will have a mean value with that same sign and hence can be interpreted as an indication of the presence of another component. Further, a disturbance in the form of an isolated disturbance will provide contributions of alternating signs to a sequence of differences of order r. Thus the reasonable interpretation of the sequence of similar signs is the presence of a polynomial contribution a_r to the D_{ri}'s.

Note that values of a_r which are small with respect to the noise components of the D_{ri}'s (i.e., small in comparison to σ_{N_r}) can fail to cause the sequence of D_{ri}'s to have the sign of a_r since a_r will no longer dominate the n_{ri}'s. Thus the absence of a sequence of D_{ri}'s of the same sign can not be taken as an indication that the polynomial component has degree less than r. However, the presence of a sequence of differences of order r having the same sign should be considered as an indication that the polynomial component will be of degree of at least r.

The nature of the property to be used for identification of appropriate polynomial degree can, perhaps, best be illustrated by a situation in which a polynomial of degree one ($P_1(t) = a_0 + a_1 t$) is fitted by the method of least squares to a set of data with a polynomial component of degree two (a parabola $P_2(t)$) and a small noise component. The situation might appear as sketched below.



The residuals errors $e_i = x_i = P_1(t_i)$ have sequences of similar signs (a sequence of negative signs, followed by a sequence of positive signs, and ending with another sequence of negative signs). Fitting a polynomial of degree two to the same data should produce a polynomial very close to $P_2(t)$ and with residuals close to the noise components and hence with signs similar to the signs of the noise components which are random.

The question of how long a sequence of D_{r_i}'s of the same sign is required to indicate the presence of a polynomial term a_r has not been resolved. For any N_{r_i} the probability that N_{r_i} is greater than zero is 0.5. The probability that a sequence of positive values for k such independent variables is the probability that a positive value will be followed by k-1 positive values is

$$P(k \text{ positive values}) = (0.5)^{k-1} ,$$

and

$$P(k \geq 5) = 1 - P(k < s) = (0.5)^{s-2} .$$

Thus

$$P(k \geq 4) = 0.125, \quad P(k \geq 5) \doteq 0.08, \quad P(k \geq 6) \doteq 0.03.$$

Thus a sequence of six or more successive differences of the same sign would be unlikely to occur do to noise alone, *if the noise components were independent.* But the noise components are not independent and, as established in Section 2.D are negatively correlated. The probability $P(k \geq 5)$ is substantially less than the value given above in the case of independence and it is suspected that a sequence of four differences of order k can be taken as an indication that the polynomial component is at least of degree k.

The situation is complicated even further by the fact that, for example, fourth order differences involve only five consecutive observations but the contemplated length of data segments considered for curve fitting is seven or eleven. It is conceivable that a polynomial fitted to the five points covered by a fourth order successive difference would be of a lower degree than one fitted to a longer sequence. On the other hand, if a polynomial of specified degree does not fit a sample of given length very well, it cannot be expected to fit a sample of greater length very well. Thus the information obtained is of a negative form in that it can be used to eliminate lower degree polynomials from further consideration.

There is a temptation to apply standard sign tests or the theory of runs to sequences of successive differences. These, however, require independence of noise components and would involve substantially more development to make them suitable for incorporation. They could be useful in the curve-fitting portion of the data smoothing program to test whether the polynomial degree is appropriate by testing whether the residual errors are of random sign or whether sign patterns exist as illustrated above.

3. APPLICATION OF SUCCESSIVE DIFFERENCES

The use of successive differences in locating outliers and in giving indication of appropriate polynomial degree for curve fitting will be illustrated for a specific set of 3-D data. This data was obtained from a test in which a torpedo was launched against a submarine at the Naval Undersea Warfare Station. The 3-D data involves coordinates recorded at equally spaced times with very few data points missing. Data for the x and y coordinates and a plot containing every fifth time is provided in the Appendix.

Suppose, now, that a noise standard deviation value $\sigma = 4$ is appropriate so that the threshold level for the fourth order differences is $D_4^* = 25.1\sigma = 100.4$. The first threshold crossings in the data occur at $t_i = 908, 909, 910, 911$. Table 5.1 shows the values of x_i, y_i and the successive differences in the neighborhood of these points. (These are reproduced here from the appendix for comparison with the results of treatment.) The situation here is somewhat confused. It does not conform to the signature (pattern) for a single isolated disturbance. One possibility procedure is to declare all four observations on x and on y as outliers. Instead of doing this consider one point at a time. Since the largest magnitudes of the D_{4i}'s occur at time $t_i = 909$, the corresponding values of x_i and y_i will be declared outliers.

Replacing these values with the average of the values at $t_i = 908$ and $t_i = 910$ yields the modified results presented in Table 5.2. All of the fourth order successive differences are now less than D_4^* and, moreover, are less than the modified thresholds given in Table 4.2 (see Figure 4.1).

There may, and should, be some doubt as to whether declaration of the observations at $t_i = 909$ as isolated outliers as sufficient treatment for this situation. As can be seen in Table 5.2, the fourth order deviations at $t_i = 911$ are quite large even though they do not exceed their threshold. Further, the signatures at both x_i and y_i are similar to what would be anticipated for isolated disturbances at $t = 911$. If, for example, the noise standard deviations were $\sigma = 3$ instead of $\sigma = 4$, then the x_i and y_i at $t = 911$ would both exceed their thresholds and be declared outliers. The results of this treatment are shown in Table 5.3. All of the large successive differences have been reduced substantially and the situation now appears to be free of disturbances. (Reduced thresholds for situations involving two disturbances separated by a non-disturbed observation are not available but should be derived so that the treatment could be completed.)

As a peripheral examination of this situation, the possibility that the observations at $t = 910$ as the initial outlier was examined. Note that the fourth order differences at $t = 909$ and $t = 910$ are reasonably close and could,

TABLE 5.1. SUCCESSIVE DIFFERENCES NEAR $t_i = 909$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
905	23983.1		-11.9		11.8	-1897.6		6.3		- 13.3
		- 48.9		12.4			-83.2		- 6.5	
906	23934.2		0.5		- 27.1	-1980.8		- 0.2		15.2
		- 48.4		-14.7			-83.4		8.7	
907	23885.8		-14.2		49.3	-2064.2		8.5		- 36.7
		- 62.6		34.6			-74.9		-28.0	
908	23823.2		20.4		-114.8	-2139.1		-19.5		110.3
		- 42.2		-80.2			-94.4		82.3	
909	23781.0		-59.8		173.2	-2233.5		62.8		-178.6
		-102.0		93.0			-31.6		-96.3	
910	23679.0		33.2		-157.4	-2265.1		-33.5		168.9
		- 68.8		-64.4			-65.1		73.6	
911	23610.2		-31.2		114.0	-2330.2		40.1		-127.8
		-100.0		49.6			-25.0		-54.2	
912	23510.2		18.4		- 78.0	-2355.2		-14.1		76.6
		- 81.6		-28.4			-39.1		22.4	
913	23428.6		-10.0		31.3	-2394.3		8.3		- 24.0
		- 91.6		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4

TABLE 5.2. OUTLIER REPLACED AT $t_i = 909$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
905	23983.1		-11.9		11.8	-1897.6		6.3		-13.3
		- 48.9		12.4			-83.2		- 6.5	
906	23924.2		0.5		-27.1	-1980.8		- 0.2		15.2
		- 48.4		-14.7			-83.4		8.7	
907	23885.8		-14.2		19.4	-2064.2		8.5		- 5.3
		- 62.6		4.7			-74.9		3.4	
908	23823.2		- 9.5		4.8	-2139.1		11.9		-15.3
		- 72.1		9.5			-63.0		-11.9	
909	23751.1*		0		- 6.2	2202.1*		0		9.8
		- 72.1		3.3			-63.0		- 2.1	
910	23679.0		3.3		-37.8	-2265.1		- 2.1		44.3
		- 68.8		-34.5			-65.1		42.2	
911	23610.2		-31.2		84.1	-2330.2		40.0		-96.2
		-100		49.6			-25.0		-54.2	
912	23510.2		18.4		-78.0	-2355.2		-14.1		76.6
		- 81.6		-28.4			-39.1		22.4	
913	23428.2		-10.0		31.3	-2394.3		8.3		-24.0
		- 91.6		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4

TABLE 5.3. OUTLIERS REPLACED AT t_i = 909, 911

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
905	23983.1		-11.9		11.8	-1897.6		6.3		-13.3
		-48.9		12.4			-83.2		- 6.5	
906	23934.2		0.5		-27.1	-1980.8		- 0.2		15.2
		-48.4		-14.7			-79.9		8.7	
907	23885.8		-14.2		19.4	-2064.2		8.5		- 5.3
		-62.6		4.7			-74.9		3.4	
908	23823.2		- 9.5		4.8	-2139.1		11.9		-15.3
		-72.1		9.5			-63.0		-11.9	
909	23751.1*		0		- 1.8	-2202.1*		0		9.8
		-72.1		7.7			-63.0		- 2.1	
910	23679.0		7.7		-15.4	-2265.1		-18.0		20.1
		-64.4		- 7.7			-45.0		18.0	
911	23574.6*		0		- 9.5	-2310.1*		- 0.1		-11.8
		-64.4		-17.2			-45.1		6.1	
912	23510.2		-17.2		24.4	-2355.2		6.0		- 3.8
		-81.6		7.2			-39.1		2.3	
913	23428.6		-10.0		- 4.3	-2394.3		8.3		- 3.9
		-96.1		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4

possibly have been reversed in order of magnitude by the noise components. The results are presented in Table 5.4. Both x_i and y_i at $t = 909$ are now indicated as outliers, exceeding not only the modified thresholds but the general threshold $D_4^* = 100.4$. Replacing both points as outliers yields the results shown in Table 5.5. An interesting outcome should be noted. The fourth order differences for both x and y at $t = 910$ now exceed the modified threshold appropriate for situations involving adjacent missing points, namely, $D_{4,910}^* = 5.1\sigma = 20.4$. (See Table 4.4.) But the observations at $t = 910$ have already been modified. This suggests that the observations at $t = 910$ should not have been considered outliers initially.

The situation in the vicinity of $t = 910$ in the data provides illustration of several features of the use of successive differences in identification of outliers. First, identification of outliers by successive differences can be awkward when there are several threshold crossings adjacent to each other. As can be seen in the situation with threshold crossings at times $t = 908$, 909, 910, and 911, rejection of the observations at $t = 909$ and 911 appear to be sufficient to reduce the ordered differences to magnitudes that could be produced by noise. A procedure involving rejection of one of the observations at a time starting with the largest one and recalculating the successive differences to be examined

TABLE 5.4. OUTLIER AT $t_i = 910$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
905	23983.1		-11.9		11.8	-1897.6		6.3		- 13.3
		- 48.9		12.4			-83.12		- 6.5	
906	23934.2		0.5		- 27.1	-1980.6		- 0.2		15.2
		- 48.4		-14.7			-83.14		8.7	
907	23885.8		-14.2		49.3	-2064.2		8.5		- 36.7
		- 62.6		34.6			-74.9		-28.0	
908	23823.2		20.4		- 98.2	-2139.1		-19.5		9.36
		- 42.2		-63.6			-94.4		65.6	
909	23781.0		-43.2		106.8	-2233.5		46.1		-111.8
		- 85.4		43.2			-48.3		-46.2	
910	23695.6		0		- 57.8	-2281.8*		- 0.1		69.7
		- 85.4		-14.6			-48.4		23.5	
911	23610.2		-14.6		47.6	-2330.2		23.4		- 71.0
		-110.0		33.0			-25.0		-37.5	
912	23510.2		18.4		- 61.4	-2355.2		-14.1		59.9
		- 81.6		-28.4			-39.1		22.4	
913	23428.6		-10.0		31.3	-2394.3		8.3		- 24.0
		- 91.6		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4

TABLE 5.5. OUTLIERS REPLACED AT $t_i = 909, 910$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
905	23983.1		-11.9		11.8	01897.6		6.3		-13.13
		- 48.9		12.4			-83.2		- 6.5	
906	23934.2		0.5		-27.1	-1980.6		- 0.2		15.2
		- 48.4		-14.7			-83.4		8.7	
907	23885.8		-14.2		20.5	-2064.2		8.5		- 9.6
		- 62.6		5.8			-74.9		- 0.9	
908	23823.2		- 8.4		2.6	-2139.1		7.6		- 6.7
		- 71.0		8.4			-67.3		- 7.6	
909	23752.2		0		- 8.4	-2202.8*		0		7.6
		- 71.0		0			-67.3		0	
910	23681.2		0		-29.0	-2266.5*		0		43.3
		- 71.0		-29.0			-67.3		42.3	
911	23610.2		-29.0		76.4	-2330.2		42.3		14.1
		-100.0		42.4			-25.0		56.4	
912	23510.2		18.4		-75.8	-2355.2		-14.1		-34.0
		- 81.6		-28.4			-34.1		72.4	
913	23428.6		-10.0		31.3	-2394.3		8.3		-24.0
		- 91.6		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4

for other threshold crossings seems reasonable. If several of the successive differences have nearly the same magnitudes, however, this could lead to rejection of the wrong observations, again, as demonstrated by rejecting the observations at $t = 910$ first.

The second feature of this example is an outgrowth of the first. An algorithm, and the subsequent computer program, which will provide satisfactory treatment for multiple adjacent threshold crossings will be awkward to produce. Nevertheless, merely identifying such situations and relegating them for manual processing should be avoided since it contradicts the objective of complete automatic processing.

The third feature arises when the first order differences are examined. There appears to be a substantial change in velocity (the a_1 term of the polynomial component) in both the x and y coordinates. The possibility of the perturbation in the vicinity of $t = 909$ being due to a change in the polynomial component instead of, or in addition to, disturbances causing outliers should be considered. This situation should be re-examined when curve-fitting to the data is attempted.

One final comment on this situation! The analysis was performed by consideration of the fourth order differences (the D_{4i}'s) only. It appears that the second and third order differences confirm the indications of the D_{4i}'s but add little of a supplementary nature. Again, this points to the use of only one order of differences for indication of outliers and the preference should be for the higher order as containing the least contamination by any polynomial component in the observations.

Another example of a threshold crossing occurs at $t = 851$ (Table 6.1). Note that in this situation only the y coordinate produces a crossing. The question as to whether the observation at x should also be rejected must be considered. In order to answer this question it may be necessary to examine the data collection process (e.g., the sensors and the geometry of the situation). The results of replacing both the x and y observations at $t = 851$ are presented in Table 6.2. Whether the improvement in the x coordinates is worth the effort is debatable at this stage.

A third event of threshold crossings in the data occurs in the vicinity of $t = 893$. Again, multiple, adjacent crossings occur but only in the y coordinates. (See Table 7.1.) The successive differences after replacing the observations at $t = 893$ are shown in Table 7.2 and after replacing the observations at both $t = 893$ and $t = 890$ in Table 7.3. Although the D_{4i}'s are well below the general bound $D_4^* = 25.1\sigma$ for $\sigma = 4$ or $\sigma = 3$, they exceed the modified bounds given in Table 4.1 for observations in the vicinity of a single missing point. This situation has not been pursued further. As in the two situations already discussed (vicinities of $t = 851$ and $t = 909$), there appears to be a substantial change in the velocity components of the vehicular path as evidenced by the values of the D_{1i}'s.

TABLE 6.1. SUCCESSIVE DIFFERENCES IN VICINITY OF $t = 851$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
848	22949.3		1.9		0.5	-1364.0		1.1		2.5
		-60.0		- 2.7			74.4		- 1.5	
849	22889.3		- 0.8		12.8	-1289.6		- 0.4		- 15.6
		-60.8		10.1			74.0		-17.1	
850	22828.5		9.3		-42.4	-1215.6		-17.5		66.9
		-51.5		-32.3			56.5		49.8	
851	22770.0		-23.0		45.1	-1159.1		32.3		-113.4
		-74.5		12.8			88.8		-83.6	
852	22702.5		-10.2		-11.4	-1070.3		-51.3		98.4
		-84.7		1.4			37.5		14.8	
853	22611.8		- 8.8		16.6	-1032.8		-36.5		- 23.3
		-93.5		18.0			1.0		- 8.5	
854	22524.3		9.2		2.8	-1031.8		-45.0		- 25.1
		-84.3		30.8			-44.0		16.6	
855	22440.0		30.0		-12.4	-1075.8		-28.4		- 7.5

TABLE 6.2. OUTLIERS AT t_i = 851 REPLACED

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
848					0.5					2.5
				- 2.7					- 1.5	
849			-0.8		1.3			- 0.4		0.5
		-60.8		- 1.4			74.0		- 1.0	
850			-2.2		- 3.6			- 1.4		2.5
		-63.0		2.2			72.6		1.5	
851	22765.5*		0		-23.9	-1142.9*		0.1		-36.8
		-63.0		-21.7			72.7		-35.3	
852			-21.7		34.6			-35.2		34.0
		-84.7		12.9			37.5		- 1.3	
853			- 8.8		5.1			-36.5		- 7.2
				18.0					- 8.5	
854					2.8					25.1
855										

TABLE 7.1. SUCCESSIVE DIFFERENCES NEAR $t_i = 893$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
887	23460.5		- 5.0		10.8	- 460.6		- 7.6		12.8
		95.5		9.1			- 42.0		13.1	
888	23546.0		4.1		-33.3	- 502.6		5.5		- 54.0
		87.6		-24.2			- 36.5		- 40.9	
889	23635.6		-20.1		43.4	- 539.1		-35.4		92.2
		69.5		18.2			- 71.9		51.3	
890	23705.1		- 0.9		-15.9	- 611.0		15.9		- 95.4
		68.6		3.3			- 56.0		- 44.1	
891	23773.7		2.4		-33.8	- 667.0		-28.2		127.9
		71.0		-30.5			- 84.2		83.8	
892	23844.7		-28.1		74.1	- 751.2		55.2		-213.7
		42.9		43.6			- 28.6		-129.9	
893*	23887.6		15.5		73.9	- 779.8		-74.3		225.3
		58.4		-30.3			-102.9		95.4	
894	23946.0		-14.8		37.1	- 882.7		21.1		-130.7
		43.6		6.8			- 81.8		- 35.3	
895	23989.6		- 8.0		6.4	- 964.5		-14.2		46.4
		35.6		13.2			- 96.0		11.1	
895	24025.2		5.2		-43.6	-1060.5		- 3.1		7.0

TABLE 7.2. OUTLIERS REPLACED AT $t_i = 893$

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
889										92.2
890					-15.9					-95.4
				3.3					-44.1	
891			2.4		-26.1			-28.2		90.8
		71.0		-22.8			-84.2		46.7	
892	23844.7		-20.4		43.3	-751.2		18.5		-65.3
		50.6		20.5			-65.7		-18.6	
893	13895.3*		0.1		-27.7	-817.0*		- 0.1		2.7
		50.7		- 7.2			-65.8		-15.9	
894	23946.0		- 7.1		6.3	-882.7		-16.0		17.7
		43.6		- 0.9			-81.8		1.8	
895			- 8.0		14.1			-14.2		9.3
				13.2					11.1	
896					-43.6					7.0

TABLE 7.3. OUTLIERS REPLACED AT $t_i = 890, 893$

t_i	x_i	D_{i1}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
887					10.8					12.8
				9.1					13.1	
888			4.1		-33.8			5.5		-46.0
		89.6		-24.7			-36.5		-32.9	
889	23635.6		-20.6		45.4	-539.1		-27.4		60.2
		69.0		20.7			-63.9		27.3	
890	23704.6*		0.1		-19.9	-603.0*		- 0.1		-47.4
		69.1		0.8			-64.0		-20.1	
891	23773.7		0.9		-22.3	-667.0		-20.2		58.8
		71.0		-21.5			-84.2		38.7	
892	23844.7		-20.4		42.0	-751.2		18.5		-57.3
		50.6		20.5			-65.7		-18.6	
893	23895.3*		0.1		-27.7	-817.0*		- 0.1		2.7
		50.7		- 7.2			-65.8		-15.9	
894	23946.0		- 7.1		6.3	-882.7		-16.0		17.7
		43.6		- 0.9			-81.8		1.8	
895			- 8.0		14.1			-14.2		9.3
				13.2					11.1	
896					-43.6					7.8

The three situations examined above are the only ones in which values of D_{4i}'s exceed the threshold $D_4^* = 25.1\sigma = 100.4$ with $\sigma = 4$. In all three situations the values of the D_{1i}'s indicate that there is a possibility of a perturbation in the form of a change in the polynomial component of the observations. It would thus appear desirable to postpone further screening for outliers until the curve-fitting portion of the data smoothing effort. After such treatment of this data set and, possibly, experience gained from examination of other data sets, the desirability of finer screening for outliers using successive differences should be reassessed.

The final comments on the data set considered here pertains to information provided by successive differences on the appropriate degree of the polynomial to be used in curve fitting. As described in Section 2.I, the primary evidence to be considered here is the existence of sequences of successive differences of a given order having the same sign. Naturally, sequences of D_{1i}'s having the same sign occur in the data and would be expected for a torpedo path since a torpedo without a velocity cannot hope to intercept its target. No attempt to fit a polynomial of degree less than one is contemplated. The only occurrences of sequences of D_{3i}'s or D_{4i}'s with the same signs and having length greater than four start at $t = 859$ and $t = 863$. Since the probability that a sequence of similar signs of length greater than $S = 4$ is $P(k \geq 5) = (0.5)^3 = 0.167$ (if the differences were due to noise only and the noise components of the differences were independent). The reduced probability of this event, due to the lack of independence, suggests that the polynomials to fit both the x and y coordinates in the segments $t = 851$ to $t = 867$ should be of degree at least three and, more likely, four. Examination of the plot of the torpedo path shown in the appendix indicates that this is, indeed, the segment of the torpedo path where the greatest changes occurred.

4. CONCLUSIONS AND RECOMMENDATIONS

During the process of model development and its subsequent application to data from a torpedo path it should be evident that successive differences provide some capability for detection of outliers. For practical purposes, an 'outlier' can be defined as an observation whose magnitude is unreasonably large when only its polynomial and noise components are considered. An algorithm for using successive differences to detect outliers is presented in Section 2.H. In this algorithm, attention is centered on the fourth order successive differences (the D_{4i}'s) and successive differences of lower orders are ignored in screening for outliers.

As a secondary use, successive differences provide some indication of appropriate polynomial degrees for the curve-fitting portion of the data smoothing process. This information is negative in form with a substantial sequence of similar signs for successive differences of a given order providing evidence that a polynomial of degree lower than that order cannot be expected to provide an acceptable fit to the data which produced that sequence.

The outline for the algorithm presented in Section 2.H requires additional development before it can be incorporated in a data smoothing program. The primary need here is for a more thorough treatment for situations involving missing points.

Since outliers are to be identified by crossings of threshold values by successive differences and since these threshold values are specified in terms of the standard deviation σ of the noise, the selection of an appropriate value for σ is fundamental to the screening process. Potential sources for values for σ are the data gathering system and the data available from torpedo paths.

The possibility of modifying the thresholds (conceptually by using a smaller value for the coefficient of σ in Section 2.D) to remove some of the outliers identified in the subsequent curve-fitting portion of the data smoothing process should be examined. Any such outliers that can be identified by successive differences can provide substantial reductions in repetitions of curve-fitting to the affected data segments. Further, the possibility of using missing points in selecting appropriate data segments for curve-fitting will be facilitated by early identification of missing points caused by elimination of outliers. This use will be discussed in a subsequent report.

APPENDIX A

DATA FROM A TORPEDO PATH AT NUWES

The model developed in this report was applied to data collected on a specific test in which a torpedo was launched against a submarine at the Naval Undersea Warfare Engineering Station. A major part of the torpedo path is sketched in the accompanying figure and the data is listed in the table which follows. Only the x and y coordinates are included.

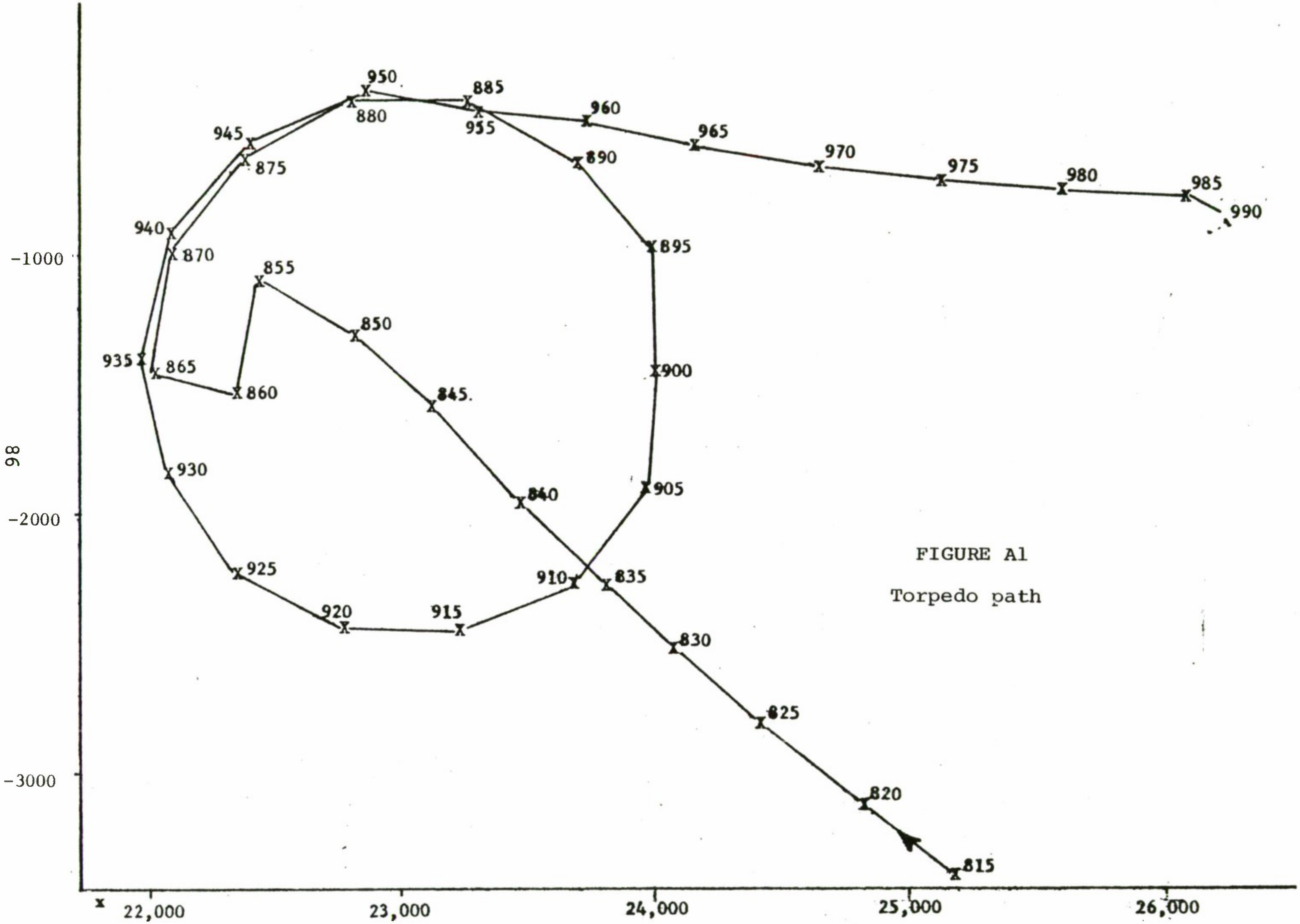


FIGURE A1

Torpedo path

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
797	26565.3		21.0		-78.6	-3802.3		- 7.7		29.3
		-42.8		-28.2			19.1		12.7	
798	26522.5		- 7.2		- 1.9	-3783.2		5.0		- 3.9
		-50.0		-30.1			24.1		8.8	
799	26472.5		-37.3		67.1	-3759.1		13.8		-54.6
		-87.3		37.0			37.9		-45.8	
800	26385.2		- 0.3		-34.4	-3721.2		-32.0		36.4
		-87.6		2.6			5.9		- 9.4	
801	26297.6		2.3		0.6	-3715.3		-41.4		47.1
		-85.3		3.2			-35.5		37.7	
802	26212.3		5.5		-21.8	-3750.8		- 3.7		-20.3
		-79.8		-18.6			-39.2		17.4	
803	26132.5		-13.1		35.8	-3790.0		13.7		-23.1
		-92.9		17.2			-25.5		- 5.7	
804	26039.6		4.1		-20.0	-3815.5		8.0		14.4
		-88.8		- 2.8			-17.5		8.7	
805	25950.8		1.3		- 9.7	-3833.0		16.7		- 4.0
		-87.5		-12.5			- 0.8		- 4.7	
806	25863.3		-11.2		34.6	-3833.8		21.4		-20.1
		-98.7		22.1			20.6		-15.4	
807	25764.6		10.9		-25.8	-3813.2		6.0		24.1
		-87.8		- 3.7			26.6		8.7	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{i1}	D_{2i}	D_{3i}	D_{4i}
808	25676.8		7.2		- 6.1	-3786.6		14.7		-16.6
		-80.6		- 9.8			41.3		- 7.9	
809	25596.2		- 2.6		21.4	-3745.3		6.8		8.6
		-83.2		11.6			48.1		0.7	
810	25513.0		9.0		-11.4	-3697.2		7.5		- 9.0
		-74.2		0.2			55.6		- 8.3	
811	25438.8		9.2		-23.0	-3641.6		- 0.8		14.9
		-65.0		-22.8			54.8		6.6	
812	25373.8		-13.6		48.4	-3586.8		5.8		-11.9
		-78.6		25.6			60.6		- 5.3	
813	25295.2		12.0		-42.8	-3526.2		0.5		1.8
		-66.6		-17.2			61.1		- 3.5	
814	25228.6		- 5.2		22.5	-3465.1		- 3.0		9.2
		-71.8		5.3			58.1		5.7	
815	25156.8		0.1		- 0.5	-3407.0		2.7		- 8.1
		-71.7		4.8			60.8		- 2.4	
816	25085.1		4.9		-17.0	-3346.2		0.3		3.9
		-66.8		-12.2			61.1		1.5	
817	25018.3		- 7.3		27.5	-3285.1		1.8		- 9.6
		-74.1		15.3			62.9		- 8.1	
818	24944.2		8.0		-34.5	-3222.2		- 6.3		17.6
		-66.1		-19.2			56.6		9.5	
819	24878.1		-11.2		38.6	-3165.6		3.2		-16.4
		-77.3		19.4			59.8		- 6.9	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
820	24800.8		8.2		-38.0	-3105.8		- 3.7		17.0
		-69.1		-18.6			56.1		10.1	
821	24731.7		-10.4		39.9	-3049.7		6.4		-22.6
		-79.5		21.3			62.5		-12.5	
822	24652.2		10.9		-36.5	-2987.2		- 6.1		22.5
		-68.6		-15.2			56.4		10.0	
823	24583.6		- 4.3		21.9	-2930.8		3.9		-14.5
		-72.9		6.7			60.3		- 4.5	
824	24510.7		2.4		-11.8	-2870.5		- 0.6		6.2
		-70.5		- 5.1			59.7		1.7	
825	24440.2		- 2.7		11.0	-2810.8		1.1		- 3.9
		-73.2		5.9			60.8		- 2.2	
826	24367.0		3.2		-10.0	-2750.0		- 1.1		7.2
		-70.0		- 4.1			59.7		5.0	
827	24297.0		- 0.9		3.4	-2690.3		3.9		-17.4
		-70.9		- 0.7			63.6		-12.4	
828	24226.1		- 1.6		5.2	-2626.7		- 8.5		25.8
		-72.5		4.5			55.1		13.4	
829	24153.6		2.9		- 4.1	-2571.6		4.9		-15.8
		-69.6		0.4			60.0		- 2.4	
830	24084.0		3.3		13.5	-2511.6		2.5		-18.3
		-66.3		13.9			62.5		-20.7	
831	24017.7		17.2		-26.0	-2449.1		-18.2		42.3
		-49.1		-12.1			44.3		21.6	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
832	23968.6		5.1		- 5.6	-2404.8		3.4		-24.7
		-44.0		-17.7			47.7		- 3.1	
833	23924.6		-12.6		36.5	-2357.1		0.3		- 2.2
		-56.6		18.8			48.0		- 5.3	
834	23868.0		6.2		-46.7	-2309.1		- 5.0		30.3
		-50.4		-27.9			43.0		25.0	
835	23817.6		-21.7		49.0	-2266.1		20.0		-46.7
		-72.1		21.1			63.0		-21.7	
836	23745.5		- 0.6		-18.4	-2203.1		- 1.7		27.3
		-72.7		2.7			61.3		5.6	
837	23672.8		2.1		2.6	-2141.8		3.9		-12.9
		-70.6		5.3			65.2		- 7.3	
838	23602.2		7.4		-17.7	-2076.6		- 3.4		18.4
		-63.2		-12.4			61.8		11.1	
839	23539.0		- 5.0		20.1	-2014.8		7.7		-21.6
		-68.2		7.7			69.5		-10.5	
840	23470.8		2.7		-13.6	-1945.3		- 2.8		22.4
		-65.5		- 5.9			66.7		11.9	
841	23405.3		- 3.2		6.9	-1878.6		9.1		-15.8
		-68.7		1.0			75.8		- 3.9	
842	23336.6		- 2.2		8.1	-1802.8		5.2		-13.0
		-70.9		9.1			81.0		-16.9	
843	23265.7		6.9		-14.4	-1721.8		-11.7		35.2
		-64.0		- 5.3			69.3		18.3	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
844	23201.7		1.6		5.0	-1652.5		6.6		-29.7
		-62.4		- 0.3			75.9		-11.4	
845	23139.3		1.3		- 6.9	-1576.6		- 4.8		13.3
		-61.1		7.2			71.1		1.9	
846	23078.2		- 5.9		18.2	-1505.5		- 2.9		6.1
		-67.0		11.0			68.2		8.0	
847	23011.2		5.1		-14.2	-1437.3		5.1		-12.0
		-61.9		- 3.2			73.3		- 4.0	
848	22949.3		1.9		0.5	-1364.8		1.1		2.5
		-60.0		- 2.7			74.4		- 1.5	
849	22889.3		- 0.8		12.8	-1289.6		- 0.4		-15.6
		-69.8		10.1			74.0		-17.1	
850	22828.5		9.3		-42.4	-1215.6		-17.5		66.9
		-51.5		-32.3			56.5		49.8	
851	22777.0		-23.0		45.1	-1159.1		32.3		-133.4
		-74.5		12.8			88.8		-83.6	
852	22702.5		-10.2		-11.4	-1070.3		-51.3		98.4
		-84.7		1.4			37.5		14.8	
853	22617.8		- 8.8		16.6	-1032.8		-36.5		-23.3
		-93.5		18.0			1.0		- 8.5	
854	22524.3		9.2		2.8	-1031.8		-45.0		25.1
		-84.3		20.8			-44.0		16.6	
855	22440.0		30.0		-12.4	-1075.8		-28.4		- 7.5
		-54.3		8.4			-72.4		9.1	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
856	22385.7		38.4		- 5.2	-1148.2		-19.3		13.0
		-15.9		3.2			-91.7		22.1	
857	22369.8		41.6		-59.9	-1239.9		2.8		-27.4
		25.7		-56.7			-88.9		- 5.3	
858	22395.5		-15.1		28.2	-1328.8		- 2.5		10.7
		10.6		-28.5			-91.4		5.4	
859	22406.1		-43.6		37.3	-1420.2		2.9		15.8
		-33.0		8.8			-88.5		21.2	
860	22373.1		-34.8		4.6	-1508.7		24.1		- 5.9
		-67.8		13.4			-64.4		15.3	
861	22305.3		-21.4		5.8	-1573.1		39.4		-12.9
		-89.2		19.2			-25.0		2.4	
862	22216.1		- 2.2		- 1.3	-1598.1		41.8		- 7.3
		-91.4		17.9			16.8		- 4.9	
863	22124.7		15.7		- 0.1	-1581.3		36.9		- 2.1
		-75.7		17.8			53.7		- 7.0	
864	22049.0		33.5		-13.6	-1527.6		29.9		-13.3
		-42.2		4.2			83.6		-20.3	
865	22006.8		37.7		-27.7	-1444.0		9.6		12.8
		- 4.5		-23.5			93.2		- 7.5	
866	22002.3		14.2		17.6	-1350.8		2.1		4.3
		9.7		- 5.9			95.3		- 3.2	
867	22012.0		8.3		5.7	-1255.5		- 1.1		1.2
		18.0		- 0.2			94.2		- 2.0	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
868	22030.0		8.1		0.9	-1161.3		- 3.1		2.1
		26.1		0.7			91.1		0.1	
869	22056.1		8.8		- 1.2	-1070.2		- 3.0		1.8
		34.9		- 0.5			88.1		1.9	
870	22091.0		8.3		- 2.0	- 982.1		- 1.1		- 8.7
		43.2		- 2.5			87.0		- 6.8	
871	22134.2		5.8		6.3	- 895.1		- 7.9		-13.3
		49.0		3.8			79.1		- 6.5	
872	22183.2		9.6		- 8.3	- 816.0		- 1.4		-14.3
		58.6		- 4.5			77.7		- 7.8	
873	22241.8		5.1		7.3	- 738.3		- 9.2		14.1
		63.7		2.8			68.5		6.3	
874	22305.5		7.9		- 7.8	- 669.8		- 2.9		-13.4
		71.6		- 5.0			65.6		- 7.1	
875	22377.1		2.9		9.8	- 604.2		-10.0		12.3
		74.5		4.8			55.6		5.2	
876	22451.6		7.7		- 8.9	- 548.6		- 4.8		- 8.0
		82.2		- 4.1			50.8		- 2.8	
877	22533.8		3.6		2.8	- 497.8		- 7.6		2.0
		85.8		- 1.3			43.2		-0.8	
878	22619.6		2.3		2.3	- 454.6		- 8.4		1.2
		88.1		1.5			34.8		0.4	
879	22707.7		3.8		- 5.2	- 419.8		- 8.0		- 0.8
		91.9		- 3.7			26.8		- 0.4	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
880	22799.6		0.1		7.3	- 393.0		- 8.4		- 1.8
		92.0		3.6			18.4		- 2.2	
881	22891.6		3.7		- 7.1	- 374.6		-10.6		5.7
		95.7		- 3.5			7.8		3.5	
882	22987.3		0.2		1.7	- 366.8		7.1		- 7.5
		95.9		- 1.8			0.7		- 4.0	
883	23083.2		- 1.6		7.8	- 366.1		-11.1		2.3
		94.3		6.0			-10.4		- 1.7	
884	23177.5		4.4		-15.3	- 376.5		-12.8		11.2
		98.7		- 9.3			-23.2		9.5	
885	23276.2		- 4.9		10.9	- 399.7		- 3.3		-14.1
		93.8		1.6			-26.5		- 4.6	
886	23370.0		- 3.3		- 3.3	- 426.2		- 7.9		4.9
		90.5		- 1.7			-34.4		0.3	
887	23460.5		- 5.0		10.8	- 460.6		- 7.6		12.8
		85.5		9.1			-42.0		13.1	
888	23546.0		4.1		-33.3	- 502.6		5.5		-54.0
		89.6		-24.2			-36.5		-40.9	
889	23635.6		-20.1		43.4	- 539.1		-35.4		92.2
		69.5		19.2			-71.9		51.3	
890	23705.1		- 0.9		-15.9	- 611.0		15.9		-95.4
		68.6		3.3			-56.0		-44.1	
891	23773.7		2.4		-33.8	- 667.0		-28.2		127.9
		71.0		-30.5			-84.2		83.8	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
892	23844.7		-28.1		74.1	- 751.2		55.6		-213.7
		42.9		43.6			- 28.6		-129.9	
893	23887.6		15.5		-73.9	- 779.8		-74.3		225.3
		58.4		-30.3			-102.9		95.4	
894	23946.0		-14.8		37.1	- 882.7		21.1		-130.7
		43.6		6.8			- 81.8		- 35.3	
895	23989.6		- 8.0		6.4	- 964.5		-14.2		46.4
		35.6		13.2			- 96.0		11.1	
896	24025.2		5.2		-43.6	-1060.5		- 3.1		7.0
		40.8		-30.4			- 99.1		18.1	
897	24066.0		-25.2		42.1	-1159.6		15.0		- 46.6
		15.6		11.7			- 84.1		-28.5	
893	24081.6		-13.5		14.2	-1243.7		-13.5		43.2
		2.1		25.9			- 97.6		-14.7	
899	24083.7		12.4		-55.7	-1341.3		1.2		-19.4
		14.5		-29.8			- 96.4		- 4.7	
900	24098.2		-17.4		34.5	-1437.7		- 3.5		12.0
		- 2.9		4.7			- 99.9		7.3	
901	24095.3		-12.7		-11.5	-1537.6		3.8		- 0.4
		-15.6		- 6.8			- 96.1		6.9	
902	24079.7		-19.5		36.9	-1633.7		10.7		-21.2
		-35.1		30.1			- 85.4		-14.3	
903	24044.6		10.6		-53.2	-1719.1		3.6		17.4
		-24.5		-23.1			- 89.0		3.7	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
904	24020.1		-12.5		23.7	-1808.1		- 0.5		- 3.7
		-37.0		0.6			-89.5		6.8	
905	23983.1		-11.9		11.8	-1897.6		6.3		-13.3
		-48.9		12.4			-83.2		- 6.5	
906	23934.2		0.5		-27.1	-1980.8		- 0.2		15.2
		-48.4		-14.7			-83.4		8.7	
907	23885.8		-14.2		49.3	-2064.2		8.5		-36.7
		-62.6		34.6			-74.9		- 23.0	
908	23823.2		20.4		-114.8	-2139.1		-19.5		110.3
		-42.2		-80.2			-94.4		82.3	
909	23781.0		-59.8		173.2	-2233.5		62.8		-173.6
		-102.0		93.0			-31.6		- 96.3	
910	23679.0		33.2		-157.4	-2265.1		-33.5		169.9
		-68.8		-64.4			-65.1		73.6	
911	23610.2		-31.2		114.0	-2330.2		40.1		-127.8
		-100.0		49.6			-25.0		- 54.2	
912	23510.2		18.4		- 78.0	-2355.2		-14.1		76.6
		-81.6		-28.4			-39.1		22.4	
913	23428.6		-10.0		31.3	-2394.3		8.3		-24.0
		-91.6		2.9			-30.8		- 1.6	
914	23337.0		- 7.1		12.2	-2425.1		6.7		6.4
		-98.7		15.1			-24.1		4.8	
915	23238.3		8.0		-29.9	-2449.2		11.5		- 9.6
		-90.7		-14.8			-12.6		- 4.8	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
916	23147.6		- 6.8		16.8	-2461.8		6.7		6.4
		- 97.5		2.9			- 5.9		1.6	
917	23050.1		- 4.8		12.1	-2467.7		8.3		2.2
		-102.3		14.1			2.4		3.8	
913	22947.8		9.3		-25.4	-2465.8		12.1		-13.1
		- 93.0		-11.3			14.5		-9.3	
919	22854.8		- 2.0		15.6	-2450.8		2.8		19.5
		- 95.0		4.3			17.3		10.2	
920	22759.8		2.3		- 0.8	-2433.5		13.0		-15.5
		- 92.7		3.5			30.3		-5.3	
921	22667.1		5.8		- 8.4	-2403.2		7.7		4.5
		- 86.9		- 4.9			38.0			- 0.8
922	22580.2		0.9		11.5	-2365.2		6.9		2.2
		- 86.9		6.6			44.9		1.4	
923	22494.2		7.5		-12.8	-2320.3		8.3		- 4.3
		- 78.5		- 6.2			53.2		-2.9	
924	22415.7		1.3		13.6	-2267.1		5.4		6.2
		- 77.2		7.4			58.6		3.3	
925	22338.5		8.7		-12.3	-2208.5		8.7		- 9.7
		- 68.5		- 4.9			67.3		-6.4	
926	22270.0		3.8		10.5	-2141.2		2.3		15.5
		- 64.7		5.6			69.6		9.1	
927	22205.3		9.4		- 9.6	-2071.6		11.4		-21.1
		- 55.3		- 4.0			81.0		-12.0	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
928	22150.0		- 5.4		5.4	-1990.6		- 0.6		17.7
		-		1.4			80.4		5.7	
929	22100.1		6.8		0.5	-1910.2		5.1		- 5.8
		-		1.9			85.5		- 0.1	
930	22057.0		8.7		- 1.5	-1824.7		5.0		- 3.3
		-		0.4			90.5		- 3.4	
931	22022.6		9.1		- 1.5	-1734.2		1.6		3.5
		-25.3		- 1.1			92.1		0.1	
932	21997.3		8.0		2.5	-1642.1		1.7		0.7
		-17.3		1.4			93.8		0.8	
933	21980.0		9.4		- 4.4	-1548.3		2.5		- 5.3
		- 7.9		- 3.0			96.3		- 4.5	
934	21972.1		6.4		6.7	-1452.0		- 2.0		7.3
		- 1.5		3.7			94.3		2.8	
935	21970.6		10.1		- 4.8	-1357.7		0.3		- 3.4
		8.6		- 1.1			95.1		- 0.6	
936	21979.2		9.0		0.4	-1262.6		0.2		- 3.5
		17.6		- 0.7			95.3		- 4.1	
937	21996.8		8.3		0.0	-1167.3		- 3.9		7.9
		25.9		0.2			91.4		3.3	
938	22022.7		8.5		- 1.5	-1075.9		- 0.1		- 9.2
		34.4		- 1.3			91.3		- 5.4	
939	22057.1		7.2		1.1	- 984.6		- 5.5		6.9
		41.6		- 0.2			85.8		1.5	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
940	22098.7		7.0		1.4	- 898.8		- 4.0		- 2.1
		48.6		1.2			81.8		- 0.6	
941	22147.3		8.2		- 2.8	- 817.0		- 4.6		- 2.3
		56.8		- 1.6			77.2		- 2.9	
942	22204.1		6.6		1.8	- 739.8		- 7.5		9.6
		63.4		0.2			69.7		6.7	
943	22267.5		6.8		- 2.1	- 670.1		- 0.8		-18.4
		70.2		- 1.9			68.9		-11.7	
944	22337.7		4.9		3.8	- 601.2		-12.5		21.0
		75.1		1.9			56.4		9.3	
945	22412.8		6.8		- 6.6	- 544.8		- 3.2		-15.4
		81.9		- 4.9			53.2		- 6.1	
946	22494.7		2.1		7.4	- 491.6		- 9.3		8.4
		84.0		2.7			43.9		2.3	
947	22578.7		4.8		- 5.0	- 447.7		- 7.0		- 4.7
		88.8		- 2.3			36.9		- 2.4	
948	22667.5		2.5		1.7	- 410.8		- 9.4		5.1
		91.3		- 0.6			27.5		2.7	
949	22758.8		1.9		1.1	- 383.3		- 6.7		-13.9
		93.2		0.5			20.8		-11.2	
950	22852.0		2.4		- 4.1	- 362.5		-17.9		24.7
		95.5		- 3.6			2.9		13.5	
951	22947.6		- 1.2		1.0	- 359.6		- 4.4		-23.5
		94.4		- 2.6			- 1.5		-10.0	

t_i	x_i	D_{i1}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
952	23042.0		- 3.8		5.3	- 361.1		-14.4		19.1
		90.6		2.7			-15.9		9.1	
953	23132.6		- 1.1		2.9	- 377.0		- 5.3		2.9
		89.5		- 0.2			-21.2		12.0	
954	23222.1		- 1.3		8.2	- 398.2		6.7		-21.6
		88.2		8.0			-14.5		- 9.6	
955	23310.3		6.7		22.7	- 412.7		- 2.9		12.5
		94.9		-14.7			-17.4		2.9	
956	23405.2		- 8.0		22.7	- 430.1		0.0		- 1.2
		86.9		8.0			-17.4		1.7	
957	23492.1		- 0.0		9.8	- 447.5		1.7		2.3
		86.9		- 1.8			-15.7		4.0	
958	23579.0		- 1.8		4.7	- 463.2		5.7		-24.0
		85.1		2.9			-10.0		-20.0	
959	23664.1		1.1		0.8	- 473.2		-14.3		46.5
		86.2		3.7			-24.3		26.5	
960	23750.3		4.8		-8.1	- 497.5		-12.2		-47.7
		91.0		- 4.4			-12.1		-21.2	
961	23841.3		0.4		3.6	- 509.6		9.0		41.2
		91.4		- 0.8			-21.1		20.0	
962	23932.7		- 0.4		0.1	- 530.7		11.0		-36.3
		91.0		- 0.7			-10.1			
963	24023.7		- 1.1		4.5	- 540.8		- 5.3		33.7
		89.9		3.8			-15.4			17.4

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
964	24113.6		2.7		- 4.0	- 556.2		12.1		-37.4
		92.6		- 0.2			- 3.3		-20.0	
965	24206.2		2.5		- 7.1	- 559.5		7.9		25.0
		95.1		- 7.3			-11.2		5.0	
966	24301.3		- 4.8		15.5	- 570.7		- 2.9		-13.4
		90.3		8.2			-14.1		- 8.4	
967	24391.6		3.4		-15.1	- 584.8		-11.3		23.2
		93.7		- 6.9			-25.4		14.8	
968	24485.3		- 3.5		16.3	- 610.2		3.5		-12.4
		90.2		9.4			-21.9		2.4	
969	24575.5		5.9		-16.7	- 632.1		5.9		- 1.7
		96.1		- 7.3			-16.0		0.7	
970	24671.6		- 1.4		17.0	- 648.1		6.6		-12.7
		94.7		9.7			- 9.4		-12.0	
971	24766.3		8.3		-38.1	- 657.5		- 5.4		24.5
		103.0		-28.4			-14.8		12.5	
972	24869.3		-20.1		61.0	- 672.3		7.1		-26.2
		82.9		32.6			- 7.7		-13.7	
973	24952.2		12.5		-44.6	- 680.0		- 6.6		22.8
		95.4		12.0			-14.3		9.1	
974	25947.6		0.5		9.9	- 694.3		2.5		- 9.2
		95.9		- 2.1			-11.8		- 0.1	
975	25143.5		-1.6		5.3	- 706.1		2.4		- 0.9
		94.3		3.2			- 9.4		- 1.0	

t_i	x_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}	y_i	D_{1i}	D_{2i}	D_{3i}	D_{4i}
976	25237.8		1.6		- 4.3	- 715.3		1.4		1.5
		95.9		- 1.1			- 8.0		0.5	
977	25333.7		0.5		- 3.2	- 723.5		1.9		- 4.8
		96.4		- 4.3			- 6.1		- 4.3	
978	25439.1		- 3.8		12.0	- 729.6		- 2.4		9.8
		92.6		7.7			- 8.5		5.5	
979	25522.7		3.9		-12.8	- 738.1		3.1		-10.5
		96.5		- 5.1			- 5.4		- 5.0	
980	25619.2		- 1.2		7.2	- 743.5		- 1.9		8.3
		95.3		2.1			- 7.3		3.3	
981	25714.5		0.9		- 2.8	- 750.8		1.4		- 4.8
		96.2		- 0.7			- 5.9		- 1.5	
982	25810.7		0.2		- 3.6	- 756.7		- 0.1		4.9
		96.4		- 4.3			- 6.0		3.4	
983	25907.1		- 4.1		8.3	- 762.7		3.3		- 6.7
		92.3		4.1			- 2.7		- 3.3	
984*	25999.5		0.0		-51.9	- 765.4		- 0.0		- 8.5
		92.4		-47.9			- 2.7		-11.8	
985	26091.8		-47.9		95.8	- 768.1		-11.8		23.6
		44.5		47.9			-14.5		11.8	
986*	26136.3		0.0		-47.9	- 782.6		0.0		-11.8
		44.5		0.0			-14.5		- 0.0	

REFERENCE

Tysver, J. B., Smoothing 3-D Data for Torpedo Paths, Naval Postgraduate School Technical Report, NPS55-78-036Pr May 1978, 64p.

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